Various Aspects Of Positive Maps In Quantum Information Theory

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Chapter 1

Introduction

The study of operator algebra is one of the active areas of research in mathematics. The concept of C^*-algebra provides a platform to study the bounded operators on Hilbert spaces in an abstract framework. Positive maps between C^*-algebras are special, as such maps preserve the structure of the cone of positive operators [1]. Historically the study of positive maps between C^*-algebras was initiated by Kadison [2, 3]. Few year later the idea of complete positive maps along with its famous dilation theorem was proposed by W. F. Stinespring [4]. It was known that the completely positive maps and completely co-positive maps form convex cones inside the cone of positive maps. Hence it was a natural query to the operator algebraists that whether any positive map can be expressed in terms of these two well known positive cones. This is basically the notion of decomposability of positive maps.

On due course people got interested in studying positive maps between algebra of matrices which is an operator algebra on finite dimensional Hilbert spaces. The study is so challenging even in low dimensions, it gained huge attention in the operator algebra community. Several seminal results in this area were obtained between 1960 to 1980. In 1963 it was proved by E. Størmer [5] that all positive maps acting between $\mathcal{B}(\mathbb{C}^2)$ to $\mathcal{B}(\mathbb{C}^2)$ are decomposable where $\mathcal{B}(\mathbb{C}^2)$ stands for the operator space acting on two dimensional complex Hilbert space $\mathbb{C}^2$. The idea of Choi-Jamiolkowski isomorphism was developed [6, 7]. In 1975 the existence of a positive map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^3)$ which is the first example of indecomposable map *i.e.* which can not be expressed as algebraic sum of completely positive and completely co-positive map was proved by M. D. Choi [8]. This map is now famously known as Choi map. This left an open question about the structure of positive maps from $\mathcal{B}(\mathbb{C}^2)$ to $\mathcal{B}(\mathbb{C}^3)$. In 1976, it was shown by S. L. Woronowich [9] that all positive maps between $\mathcal{B}(\mathbb{C}^2)$ and $\mathcal{B}(\mathbb{C}^3)$ as well as between $\mathcal{B}(\mathbb{C}^3)$ and $\mathcal{B}(\mathbb{C}^2)$ are also decomposable.

Though the study of positive maps was confined within the mathematics community for a long time, it readily got the attraction to the physics community when the connection between positive map and the theory of entanglement was discovered. Quantum entanglement [10–12] is a form of quantum correlation which can be harnessed to accomplish various quantum information theoretic tasks like dense coding [13], quantum teleportation [14], quantum key distribution [15], true randomness generation [16] which are imposable in the classical world. Therefore quantum entanglement is an essential resource. Whenever we talk about a resource, the question that comes to our mind first is how to identify the resource. In this context, precisely the question is, given a quantum state how to identify whether it is entangled or separable. Answer to this question has a nice link with the theory of positive maps. In 1996 a necessary condition for separability of a bipartite mixed density matrix was identified by A. Peres [17]. This condition is given by means of transposition map which is an example of positive map. In the same year a necessary and sufficient condition for certification of bipartite quantum entanglement in terms of positive maps was established by Horodecki et al. [18]. Though the criterion is highly non trivial but it shows how the theory of positive maps is deeply connected with entanglement theory. This paper has another important significance. By exploiting Størmer's and Woronowich's result [5, 9] a relatively simple sufficient condition of entanglement detection for two qubits and qubit-qutrit systems was devised in the same paper. Thus we have got a necessary and sufficient condition of detecting entanglement for qubit-qubit and qubit-qutrit systems. This condition is based on transposition map and it is famously known as Peres-Horodecki criterion for entanglement detection. For higher dimensional systems certification of entanglement is still uncharacterised and later it has been proved that certifying whether a given unknown state is separable is a NP hard problem [19].

Gradually this paved the foundation to study of a weak form of entanglement known as positive under partial transposition (PPT) entanglement in higher dimensional systems [20]. PPT entanglement is weak as no maximally entangled state can be distilled from it under local operation and classical communication [21]. After these seminal works, both mathematics and physics community have tried to explore the structure of positive maps [22–25] and have been able to shed some light on the topic. The connection between indecomposability and PPT entanglement was explored further [26–30]. It is important here to mention that the connection between PPT entanglement and positive indecomposable maps was not a new concept. In 1982 a necessary and sufficient condition for decomposability of a positive map was proposed by E. Størmer [31]. Along with this equivalent condition, the Choi map was shown to be indecomposable in an alternative approach. The approach was based on evaluating the map on one part of certain operators. Later such operators have been identified as PPT entangled density matrix and this technique has become an useful tool to infer indecomposability of a positive map and PPT entanglement.

An experimental friendly technique to detect quantum entanglement has been developed on due course [32]. This is the idea of entanglement witness [33, 34]. Entanglement witness is an Hermitian operator which can separate an entangled state from all separable states. This is done by exploiting the geometric structure of the state space. Clearly, finding an entanglement witness corresponding to some entangled state is a non-trivial task. Again the non-triviality lies in the connection of bipartite entanglement witness with the theory of positive maps [35]. All of these discussions lead to the fact that how important it is to understand the structure of positive maps to understand bipartite entanglement theory properly. At this point it is worthwhile to mention a few examples of indecomposable maps are known in literature. A new positive indecomposable map on $\mathcal{B}(\mathbb{C}^4)$ was proposed by Robartson [36–40]. In 1988, a stricter notion of positive map called atomicity was introduced by Tanahashi and Tomiyama [41]. It was shown that the Choi map is not only indecomposable, it is even atomic. Later the Choi map was generalised as three parameter family of maps [42]. A class of positive indecopmposable maps on operators acting on d dimensional complex Hilbert space was defined by Breuer and Hall [43, 44]. It was proved by D. Chruściński et al. that Breuer-Hall map is connected with Robertson's map. The atomicity of the Robertson's map has also been proved which in turn proves the atomicity of the Breuer-Hall map [45]. A generalization of the positive map considered by Hall was proposed in [45] and the atomicity of the generalized map under certain condition has been discussed in the same paper. Despite of considerable efforts from both mathematics and physics community [46–58] the structure of positive map is still uncharacterised fully and requires more research in future. In 2015 another indecomposable positive map [59] on $\mathcal{B}(\mathbb{C}^3)$ has been proposed by M. Miller et al. This map is fundamentally different from the Choi map. It is clear from the definition that every atomic map is indecomposable, but the other way round is not true in general. Recently in 2016 it has been shown by Yang et al. that on $\mathcal{B}(\mathbb{C}^3)$ every positive indecomposable map is also atomic [60].

So far we have discussed on development of theory of positive maps from the algebraic point of view. There is also a geometric notion of positive maps. Though we have adopted algebraic approach to study positive maps in this thesis, but for sake of completeness it is important to mention some of the important as well as interesting developments of the study of positive maps through geometric ideas. As mentioned earlier, the set of positive maps forms a convex cone. Therefore the tools of convex geometry [61] can be applied to understand the structure of positive maps [62–64]. Duality relations play an important role to study the cones and hence it is important from the context of entanglement [65]. Two main questions about the positive maps from the perspective of convex geometry is regarding their extremality and exposedness [66–70]. It is well known that every exposed point in a convex set is extreme but the other way round is not true in general. Famous Straszewicz theorem tells us that exposed points form a dense subset of the set of extreme points. However it is a non-trivial task to find an extreme point of the cone of positive maps. Choi map is an example of such point [71].

Facial structures of the cone of positive maps have been studied by S. H. Kye. The connection of the theory of entanglement with these geometric notions has been explored in [72]. A sufficient condition for exposedness of positive maps has been provided by D. Chruściński et al. [73].

Though the study of positive map finds enormous applications in the theory of quantum entanglement, this is not the only aspect where the picture is so. One of the most important subclass of positive map is the class of completely positive maps [4]. In entanglement theory positive but not completely positive maps play an instrumental role. Completely positive maps can not be used to detect entanglement as they are insensitive for entanglement. But one of the crucial features of complete positive maps is that such maps are physically realisable. Complete positivity along with trace preservation provides the mathematical model for quantum channels [74, 75]. Channels are important from the aspect of transformations. Quantum channels are studied to understand the transformations of quantum states. Dynamics of a quantum system *i.e.* time evolution of a quantum state is also governed by completely positive trace preserving maps. Hence another important aspect of the theory of positive map in quantum information theory is the study of the dynamics of a quantum system. According to the quantum postulate, the dynamics of a quantum system is given by unitary evolution. But this is an ideal situation where the quantum system is closed *i.e.* it does not interact with its surroundings. In reality quantum systems are very fragile and readily interact with the corresponding environment. So in practice quantum systems are open and its actual dynamics is not governed by unitary evolution. Due to such unavoidable interactions system looses information to the environment. This one-way information flow characterized by the monotonic relaxation towards the stationary states is a direct consequence of the Born-Markov approximation [76]. Born-Markov approximation is valid for very large stationary environments which leads to Complete Positive (CP)-divisibility of the dynamics [77–79]. However, beyond Born-Markov limit, the CP-divisibility breaks down [80], triggering non-Markovian backflow of information [81–96], *i.e.* information can flow back from the environment to the system. This is a kind of memory effect whose classical counterpart is studied in classical stochastic process. Recently, it has been established that non-Markovian information backflow acts as a resource in various quantum mechanical tasks. For example, non-Markovianity allows perfect teleportation with mixed states [97], efficient entanglement distribution [98], improvement of capacity for long quantum channels [99] and efficient work extraction from an Otto cycle [100]. The exploitation of information backflow as a resource has also been considered in entangling operations and quantum metrology [101–105]. For all of these mentioned cases, non-Markovianity gets inter-converted via information backflow, into other resources like entanglement, coherent information, extractable work etc. It can also be exploited for efficient quantum control [106]. This establishes the necessity of a formal resource theory of non-Markovianity.

Non-Markovianity or memory effect in classical sense has been rigorously formulated and very well studied in classical probability theory [107–109]. To formulate the dynamics of open quantum system mathematically C^*-algebraic approach have been adopted around 1980s [110,111]. Unlike classical stochastic process, probabilities in quantum stochastic process are defined on non-commutative structures. One should keep in mind that probabilities are obtained here via quantum measurements. Interested readers can go through the review by Á. Rivas, S. F. Huelga and M. B. Plenio [77]. Quantum notion of non-Markovianity is studied by using time parametrized completely positive trace preserving maps known as dynamical maps. There is no unanimous definition of quantum non-Markovianity. It precisely depends on the context under study. The hierarchy between various definitions of non-markovianity has been studied in literature [112]. Among them there is an approach by H. P. Breuer, E. M. Laine and J. Piilo (BLP) [113] which considers the change in distinguishability of quantum states during the dynamics. If there is no backflow of information, the distinguishabilty of states decreases during the dynamics. This is a signature of Markovianity. Another approach is by Á. Rivas, S. F. Huelga and M. B. Plenio (RHP) in which the dynamics is expressed in terms of the concatenation of positive maps. This gives rise to the idea of propagator of the dynamics. Depending upon the nature of the propagator, Markovianity is inferred. This approach is known as divisibility approach [80]. CP-divisibility breaking is necessary to have information backflow. This implies whenever distinguishability of system state increases, information is coming back from the environment and the divisibility of the dynamics has been broken. There may be some divisible non-Markovian dynamics in nature but considering non-Markovian information back flow as a resource we can exclude them from our discussion. It is important to mention that the connection between BLP and RHP approach has been already established [114–116]. There is another approach to non-Markovianity which is based on master equation [117, 118]. Master equation is basically the dynamical equation of the system. This is an approach by means of differential equation of the dynamics . The relationship between time dependent Gorini-Kosakowski-Sudarshan-Lindblad (GKSL) master equation and CP-divisibility has been established [77]. Recently quantum non-Markovianity has been approached from process tensor formalism [119]. In this thesis we are interested to study the resource theoretical framework of non-Markovianity. To be precise, non-Markovian backflow of information is the resource which we want to study. As mentioned earlier that divisibility breaking of the dynamics is necessary to have the backflow of information from the environment to the system. Hence we have adopted the approach based on divisibility breaking of the dynamics to study quantum non-Markovianity. We also use Lindblad type master equation [117] to consider the structure of the propagator of the dynamics.

In this thesis we have studied both the aspects of positive maps in quantum information theory. We have studied the theory of entanglement as well as we have studied the dynamics of open quantum systems. Chapter 1 is devoted to introduction. In chapter 2 we have kept the necessary mathematical prerequisites to understand the rest of the chapters. In chapter 3 we have constructed a novel one parameter family of positive map which is indecomposable in nature. We also show its connections with the theory of entanglement. From chapter 4 onwards we concentrate in studying resource theoretic framework of quantum non-Markovianity. In chapter 4 we have constructed the convex resource theory of non-Markovianity. In chapter 5 we discuss on the witness of non-Markovianity which is an essential point from the perspective of resource theory. In chapter6 we concentrate in the question of resource interconversion. We study quantum non-Markovianity in the backdrop of other resource theories. Finally in chapter 7 we summarize our works and discuss some open questions from this thesis.

Chapter 2

Preliminaries

In this chapter we shall discuss the preliminary ideas required to understand the thesis. The discussions will be on positive maps in general and its connections with the theory of entanglement and the dynamics of open quantum system. Throughout this thesis we shall consider discrete quantum systems and hence the underlying Hilbert space is finite dimensional and the operator algebra is the algebra of matrices. Let $\mathbb{C}^d$ be the complex Hilbert spaces of dimension d. Let $\mathcal{B}\left(\mathbb{C}^d\right)$ denotes the space of all operators acting on $\mathbb{C}^d$. Recall that operators acting on finite dimensional spaces are bounded and can be represented as matrices with respect to some basis. The sub collection of $\mathcal{B}\left(\mathbb{C}^d\right)$ consisting of Hermitian, positive semi-definite operators having unit trace is known as the set of density operators acting on $\mathbb{C}^d$. Let us denote this subclass by $\mathcal{S}\big(\mathbb{C}^d\big)$. Mixed quantum states are represented by density matrices. Whereas pure quantum states are associated with unit vectors in $\mathbb{C}^d$. Throughout this thesis we shall denote vectors in $\mathbb{C}^d$ using ket notation. Commutator of two operators A and B is given by [A,B]= AB $-$ BA. Hilbert-Schmidt inner product between two operators A and B is defined as $\langle A, B\rangle =$ $\mathrm{Tr}(A^\dagger B)$. This in turn gives the notion of Hilbert-Schmidt norm given by $||A||_{HS} = \sqrt{\mathrm{Tr}(A^\dagger A)}$ for any operator A. Here Tr[.] stands for the trace of the operator. On the other hand trace norm of an operator A is given by $||A||_1 = \mathrm{Tr}\sqrt{A^\dagger A}$. Let us recall a fundamental fact that on finite dimensional normed linear space any norm is topologically equivalent. In an inner product space we can talk about positive semi-definiteness of an operator. A Hermitian operator X is said to be positive semi-definite if $\langle\psi|X|\psi\rangle \geq 0$ for every $|\psi\rangle \in \mathbb{C}^d$. A bipartite quantum system is represented by tensor product of two Hilbert spaces. Let X be an operator acting on $\mathbb{C}^d \otimes \mathbb{C}^d$ *i.e.* $X \in \mathcal{B}\left(\mathbb{C}^d \otimes \mathbb{C}^d\right)$. The notion of block positivity can be defined for an operator acting on tensor product Hilbert space. A given Hermitian operator $X \in \mathcal{B}\left(\mathbb{C}^d \otimes \mathbb{C}^d\right)$ is said to be block positive if $\langle x| \otimes \langle y| X |x\rangle \otimes |y\rangle \geq 0$ for any product vector $|x\rangle \otimes |y\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. In the following section we shall discuss about a brief overview of positive maps acting between algebra of operators.

2.1 General Overview of positive maps

We first start with the following definitions.

Definition: A linear map $\Lambda : \mathcal{B}\left(\mathbb{C}^{d_1}\right) \rightarrow \mathcal{B}\left(\mathbb{C}^{d_2}\right)$ is said to be positive if $\Lambda(X) \geq \Theta$ for any positive semi-definite $X \in \mathcal{B}\left(\mathbb{C}^{d_1}\right)$, where Θ denotes the zero operator.

Definition: A linear map is said to be k-positive if the map $\mathbb{I}_k \otimes \Lambda : \mathcal{B}\left(\mathbb{C}^k\right) \otimes \mathcal{B}\left(\mathbb{C}^{d_1}\right) \rightarrow \mathcal{B}\left(\mathbb{C}^k\right) \otimes \mathcal{B}\left(\mathbb{C}^{d_2}\right)$ is positive for some $k \in \mathbb{N}$. Here $\mathbb{I}$ stands for the identity map.

Definition: A linear map is said to be completely positive if it is k-positive for all $k \in \mathbb{N}$.

Remark 1: A liner map $\Lambda : \mathcal{B}\left(\mathbb{C}^{d_1}\right) \rightarrow \mathcal{B}\left(\mathbb{C}^{d_2}\right)$ is completely positive if it is d positive where $d = min\{d_1, d_2\}$

Remark 2: Transposition map T is a positive map but it is not completely positive.

Definition: A linear map is called trace preserving if it preserves the trace after the action of the map. Mathematically for a trace preserving map Λ, $\mathrm{Tr}[\Lambda(X)] = \mathrm{Tr}[X]$ for all operators X.

Next we shall state two important and useful representations of completely positive maps.

Choi-Kraus: [7, 120] A map $\Lambda : \mathcal{S}\left(\mathbb{C}^d\right) \to \mathcal{S}\left(\mathbb{C}^d\right)$ is linear, completely positive and trace preserving if and only if Λ has a Choi-Kraus decomposition as

$$\Lambda(X) = \sum_{i=0}^{d-1} K_i X K_i^\dagger$$

where K_is are the operators satisfying $\sum_{i=0}^{d-1} K_i^\dagger K_i = \mathbf{I}_d$, where $\mathbf{I}$ stands for the identity operator.

Stinespring: [4, 75] Let $\Lambda : \mathcal{S}\left(\mathbb{C}^d\right) \to \mathcal{S}\left(\mathbb{C}^d\right)$ be a completely positive trace preserving linear map. Then there exists a finite dimensional Hilbert space $\mathcal{E}$ and an unitary operator U acting on $\mathbb{C}^d \otimes \mathcal{E}$ such that

$$\Lambda(X) = \mathrm{Tr}_{\mathcal{E}}[U(X \otimes \sigma)U^\dagger]$$

where $\mathrm{Tr}_{\mathcal{E}}[.]$ stands for the trace over Hilbert space $\mathcal{E}$ and σ is some state on $\mathcal{E}$.

Definition: A linear map Λ is said to be k-co-positive if $\mathbb{I}_k \otimes (T \circ \Lambda)$ is positive for some $k \in \mathbb{N}$ and completely co-positive if $T \circ \Lambda$ is completely positive, where T stands for the transposition map.

Definition: A positive map Λ is known to be decomposable if it can be expressed as $\Lambda = \Lambda_1 + T \circ \Lambda_2$ where Λ_1 and Λ_2 are completely positive maps. Otherwise, it is said to be indecomposable.

Definition: A positive linear map Λ is said to be atomic if it can not be written as a sum of a 2-positive and 2-co positive map.

By definition every atomic positive map is indecomposable but the converse is not true in general.

Given a positive linear map $\Lambda : \mathcal{B}\left(\mathbb{C}^{d_1}\right) \rightarrow \mathcal{B}\left(\mathbb{C}^{d_2}\right)$, its dual map $\Lambda^\dagger : \mathcal{B}\left(\mathbb{C}^{d_2}\right) \rightarrow \mathcal{B}\left(\mathbb{C}^{d_1}\right)$ is defined as,

$$\mathrm{Tr}[\Lambda^\dagger(X)\ Y] = \mathrm{Tr}[X\ \Lambda(Y)]$$

It is to be noted that a linear map Λ is positive if and only if its dual $\Lambda^\dagger$ is positive.

Definition: A linear map is said to be Hermiticity preserving if it preserves Hermiticity of an operator after the action *i.e.* $[\Lambda(X)]^\dagger = \Lambda(X^\dagger)$ for any operator X.

Next let us discuss one of the central ideas of the theory of positive map.

Choi-Jamiołkowski (CJ) Isomorphism [6, 7]: Given any positive linear map $\Lambda : \mathcal{B}\left(\mathbb{C}^{d}\right) \rightarrow \mathcal{B}\left(\mathbb{C}^{d}\right)$ there exists an operator $\mathcal{C}^\Lambda \in \mathcal{B}\left(\mathbb{C}^{d} \otimes \mathbb{C}^{d}\right)$ which is the isomorphic image of the map Λ.

The isomorphism is given by,

$$\Lambda \mapsto \mathcal{C}^\Lambda = \mathbb{I} \otimes \Lambda\left(|\psi\rangle\langle\psi|\right)$$

where $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle \equiv \frac{1}{\sqrt{d}} \sum_i |ii\rangle$, and $i \in \{|0\rangle, |1\rangle, ..., |d-1\rangle\}$ is an orthonormal basis of $\mathbb{C}^d$.

Note: Though the CJ isomorphism can be defined for a positive linear map having different input and output dimension of the operator space, for our purpose we shall restrict to the same input and output dimension.

The main essence of CJ isomorphism is that it translates everything form the language of positive map to the language of operators. The isomorphic image of a positive map under CJ isomorphism is called its corresponding Choi operator or Choi matrix. It is important to note that

- A positive linear map $\mathit{\Lambda}$ is completely positive if and only if its Choi operator $\mathcal{C}^{\mathit{\Lambda}}$ is positive semi-definite.
- A linear map $\mathit{\Lambda}$ is positive but not completely positive if and only if its Choi operator $\mathcal{C}^{\mathit{\Lambda}}$ is block positive.
- A positive linear map $\mathit{\Lambda}$ is Hermiticity preserving if and only if its corresponding Choi $\mathcal{C}^{\mathit{\Lambda}}$ operator is Hermitian.

2.2 Entanglement detection with positive maps

Next let us discuss on the connection of positive map and the theory of entanglement. Before going to the main discussion, let us briefly illustrate the idea of bipartite entanglement. As mentioned in the introduction quantum entanglement is an essential resource for certain quantum information processing tasks. It is a form of quantum correlation defined at the level of quantum states. Quantum entanglement is defined for both pure as well as mixed quantum states. Pure state entanglement has a nice certification criterion in terms of singular value decomposition. This is known as Schmidt decomposition [74]. Unlike pure state, characterization of entanglement for mixed quantum state is quite challenging. Let us start with the definition of mixed entanglement first.

Definition: A bipartite state $\rho_{AB} \in \mathcal{B}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right)$ is said to be separable if there exists density operators $\sigma_i \in \mathcal{B}\left(\mathbb{C}^{d_1}\right)$ and $\delta_i \in \mathcal{B}\left(\mathbb{C}^{d_2}\right)$ and a probability distribution $\{p_i\}$ such that $\rho_{AB} = \sum_i p_i \sigma_i \otimes \delta_i$. Otherwise the state is entangled.

A. Peres first identified that transposition map can be used to have a necessary condition for bipartite entanglement [17]. He proved the following theorem.

Theorem 1: [17] If a bipartite density matrix ρ_{AB} is separable then it is positive semi-definite under partial transposition *i.e.* PPT. Mathematically for any separable state ρ_{AB} , $\mathbb{I} \otimes T(\rho_{AB}) \geq \mathit{\Theta}$

Exploiting the definition of separable density matrix and using the fact that transposition map carries positive semideinite operators to another positive semi-definite operator, the result follows. Eventually this implies every quantum state which is not positive under partial transposition *i.e.* NPPT is entangled. This is a remarkable discovery as it is the first evidence that positive map can be used to detect entanglement. Though the condition is necessary but not sufficient. In the same year Horodecki et. al. came up with a seminal result [18] which established a necessary and sufficient condition for certification of mixed state entanglement in terms of positive linear maps. In the following theorem we discuss the criterion.

Theorem 2: [18] A bipartite sate $\rho \in \mathcal{B}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right)$ is separable if and only if for all positive maps $\Lambda : \mathcal{B}\left(\mathbb{C}^{d_1}\right) \rightarrow \mathcal{B}\left(\mathbb{C}^{d_2}\right)$ the operator $\mathbb{I} \otimes \Lambda(\rho)$ is positive semi-definite.

Non triviality of the theorem can be understood from the statement itself. To certify a given state as separable (or entangled otherwise) one has to scan through each and every positive map acting on the corresponding operator space. This is mathematically challenging as the structure of positive maps is still not well understood even in low dimensions. Thanks to the seminal works by E. Størmer [5] and S. L. Woronowich [9] which have completely characterised the structure of positive map acting on $\mathcal{B}(\mathbb{C}^2)$ to $\mathcal{B}(\mathbb{C}^2)$ and $\mathcal{B}(\mathbb{C}^2)$ to $\mathcal{B}(\mathbb{C}^3)$ and vice versa. As a consequence of these results one can understand that transposition map is the only positive but not completely positive map acting on these dimensions. Armed with these results, Horodecki et. al. have been able to establish a sufficient criterion to certify mixed entangled states in two qubits system as well as qubit qutrit systems in terms of transposition map.

Peres-Horodecki Criterion: Let ρ_{AB} be a two qubits or qubit-qutrit mixed quantum state. Then ρ_{AB} is separable if and only if it is positive under partial transposition *i.e.* PPT.

Peres-Horodecki criterion ensures that any PPT quantum state in two qubits or qubit-qutrit system is separable as transposition map can detect any mixed entanglement in these dimensions. However this is not the case in higher dimensions. There are mixed quantum states in higher dimensions which is positive under partial transposition but entangled. Such entangled states are known as PPT entangled states. Horodecki et al. have proved the connection between PPT entanglement and entanglement distillation. The process by which maximally entangled state is distilled out of mixed entangled states under local operation and classical communication is known as entanglement distillation [121]. No maximally entangled state can be distilled from PPT entangled state. That is why PPT entangled states are termed as 'bound entangled'. Hence PPT entanglement is a weak form of entanglement which can not be detected via transposition map. This automatically gives rise the question how to detect PPT entanglement. The next theorem gives the answer to this question.

Theorem 3: [31] Positive indecomposable maps are necessary to detect PPT entanglement.

As positive decomposable maps can be written as the sum of a completely positive and completely co-positive maps, Applying the map partially on one part of an PPT state gives rise to a positive semi-definite operator. Hence A positive decomposable map can not detect a PPT entangled state.

Certification of indecomposability of a positive map via a PPT entangled state was first done by E. Størmer [31]. Interestingly there was no concept of PPT entanglement at that moment. Stormer proved the indecomposability of Choi map via this approach. It is important to mention here that there is no universal positive indecomposable map to detect all PPT entangled states. The same is true for PPT entanglement as well *i.e.* one PPT entangled state may be detected by some positive indecomposable map but may not be detected by other positive indecomposable maps. This shows how non trivial it is to find a proper positive indecomposable map to detect certain PPT entangled state.

So far we have discussed the detection of quantum entanglement via positive maps. This is the best way to detect entanglement theoretically. Unfortunately in practice things are not easy in general. Practical situation demands resource detection (here entanglement) in an experimental friendly way. In laboratory what people can measure is the observables. According to quantum postulates, observables are Hermitian operators. The idea of entanglement witness becomes handy in such situations. Next we shall discuss briefly on entanglement witness. Detailed discussion on construction, analysis and classification of entanglement witness has been provided in [35]

2.3 Entanglement witness

Let us start with the definition of entanglement witness.

Definition: A Hermitian operator $\mathcal{W}$ is said to be an entanglement witness if $\mathrm{Tr}\,(\mathcal{W}\rho) \geq 0$ for all separable state ρ and there exists at least one entangled state σ for which $\mathrm{Tr}\,(\mathcal{W}\sigma) < 0$.

being Hermitian operator, entanglement witnesses are observables by construction. Entanglement witness gives rise to non negative expectation values to every separable states and produces negative expectation value to the entangled state which it detects. Existence of entanglement witness has a nice geometric aspect. We shall now discuss about the existence of entanglement witness by exploiting fundamental results from convex geometry [61].

If one looks carefully the definition of entanglement witness, it is an inequality whose geometry is given by a hyperplane separation. Therefore existence of an entanglement witness is basically the existence of a separating hyperplane that separates an entangled state from the set of separable states. Before going to discuss about the existence of such witness, we illustrate an essential geometric and topological property of a set inside an Euclidean space. Then we shall discuss the structure of separable states in light of these properties.

Definition: In an Euclidean space a subset C is said to be convex if the convex combination of any two points inside the set A is again a member of C. Mathematically a subset A is convex if

$$\forall x, y \in C, \qquad \lambda x + (1-\lambda) y \in C \qquad \text{with} \qquad \lambda \in [0,1]$$

Examples: A triangular region or a square region or a circular region on two dimensional plane.

Remark: Geometrically a set is convex if one considers a line segment joining any two points inside the set, the line segment entirely lies within the set. Though the notion of convexity can be defined in any topological vector space. For our purpose we shall consider only Euclidean spaces.

Definition: Let C be a convex set. A point in the set is said to be an extreme point if there does not exist any non-trivial decomposition of the point in terms of other points in C. Mathematically a point $x \in C$ is an extreme point if

$$x = \lambda y + (1-\lambda)z \quad \text{with} \;\; y, z \in C \;\; \text{and} \;\; \lambda \in [0,1] \;\; \text{then either} \;\; x = y \;\; \text{or} \;\; x = z$$

Definition: A convex set is said to be a polytope if it has finitely many extreme points.

From the definition it is clear that a triangular region is a polytope but a circular region is not a polytope.

Next let us discuss briefly about compactness of a set. Compactness is a topological property of a topological space. In general it is defined in terms covers made of open sets. As we restrict ourselves in Euclidean space, compactness can be characterized by means of famous Heine-Borel theorem.

Heine-Borel Theorem: A subset inside an Euclidean space is compact if and only if it is closed and bounded.

In light of the above discussions we state the following proposition.

Proposition 1: Set of separable quantum states forms a convex and compact subset of the state space.

Convexity of the set of separable states follows from the definition of separable states. Since the state space is inside an Euclidean space (As Hermitian operators form a real vector space equipped with inner product), we have Heine-Borel theorem. It tells that a subset of an Euclidean space is compact if it is bounded and topologically closed. The set of separable states obeys both the properties and hence compact. Having a convex and compact set we have geometric Hahn-Banach theorem [61] for hyperplane separation to prove the existence of the separating hyperplane.

Theorem 4 (Hyperplane separation theorem): [61] Let A and B be two subsets of an Euclidean space. Let A be convex and compact and B be convex and closed. If $A \cap B = \emptyset$, then there exists a separating hyperplane which separates A and B.

From the previous proposition we have the convexity and compactness of separable quantum states. An entangled state, being a singleton point is always convex and closed. Therefore by virtue of the geometric Hahn-Banach theorem there always exists a separating hyperplane which separates the entangled state from the set of separable states. This shows the existence of a witness corresponding to some entangled state. It is important to note that the hyperplane separating the entangled state from separable states is not unique. Given an entangled state one can have more than one entanglement witnesses to separate it from separable states. This readily raises the question of optimal entanglement witness [34]. Geometrically optimal entanglement witness can be thought as a tangent plane corresponding to the set of separable states.

So far we have discussed about the existence of entanglement witness. Another fundamental question regarding this is how to construct an entanglement witness. Again the theory of positive linear maps play an important role in this construction. Next we shall address this point.

We have already mentioned that CJ isomorphism 2.1 carries a positive maps onto an operator. Properties of the maps get reflected through the corresponding Choi operator. We have seen that the Choi operator corresponding to a positive but not completely positive map is block positive. As a consequence of which it provides non negative expectation to any product vectors. Therefore the Choi operator corresponding to any Hermiticity preserving , positive but not completely positive map gives rise to an entanglement witness. Mathematically,

Given a Hermiticity preserving positive but not completely positive map Λ, its Choi operator given by

$$\mathcal{C}^{\Lambda} = \mathbb{I} \otimes \Lambda\left(|\psi\rangle\langle\psi|\right)$$

is an entanglement witness, where $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$, is the maximally entangled state in dimension d. Clearly CJ isomorphism depicts structural essence of entanglement witness. In the previous subsection we have defined decomposable positive maps as the algebraic sum of a completely positive map and a completely co positive map. CJ isomorphism gives rise to the idea of decomposable entanglement witness. The Choi operator corresponding to a decomposable positive map is a decomposable entanglement witness. The following proposition gives the structure of a decomposable entanglement witness

Proposition 2: [35] Let $\mathcal{W}$ be a decomposable entanglement witness. The $\mathcal{W}$ can be expressed as

$$\mathcal{W} = P + Q^T$$

where P and Q are two positive operators and T stands for transposition map. If an entanglement witness is not decomposable it is called indecomposable. Entanglement witness corresponding to an indecomposable positive map is an indecomposable entanglement witness.

In the next subsection we shall illustrate the connection between the theory of positive linear maps and the dynamics of open quantum system. Positive maps take positive operators to positive operators and hence to carry a quantum state to another quantum state positive maps are necessary. A quantum state may be entangled with another quantum state and hence we need a stricter notion of positive maps ,*viz.*, completely positive maps along with trace preservation to ensure a valid transformation of the quantum state. In the following discussion we shall focus on the time evolution of a quantum system.

2.4 Positive maps and the dynamics of open quantum system

According to the postulate of quantum theory dynamics of a closed quantum system is unitary. To model the dynamics of an open quantum system one has to consider a more general notion than the unitary evolution.

Let t_0 be the initial time of a dynamics. Then Time evolution of an open quantum system is governed by a family of completely positive trace preserving maps $\{\Lambda_{(t,t_0)}\}_{t\geq t_0}$ satisfying the following composition law,

$$\Lambda_{(t_3,t_1)} = \Lambda_{(t_3,t_2)} \circ \Lambda_{(t_2,t_1)} \tag{2.1}$$

for any $t_3 \geq t_2 \geq t_1$ and $\circ$ stands for composition of two maps.

It is clear that if dynamics is considered from the initial time t_0 to some later time t, then the dynamics is governed by complete positivity. To be precise, during the dynamics let us consider three temporal points, t_0 , t_1, t_2 among which t_0 is the initial point and $t_2 \geq t_1 \geq t_0$. Then the dynamics from t_0 to t_1 and from t_0 to t_2 are governed by complete positivity. Incidentally within the dynamics the nature of evolution between t_1 to t_2 is not specified. We can say that $\Lambda_{(t_2,t_1)}$ is a map which controls as well as characterizes the dynamics. This leads us to the following definitions.

Definition: **(Positive Divisibility)** A quantum system undergoing time evolution characterized by trace preserving linear maps $\{\Lambda_{(t_2,t_1)}, t_2 \geq t_1 \geq t_0\}$ is positive divisible or P-divisible if for every t_2 and t_1, $\Lambda_{(t_2,t_1)}$ is a positive map and fulfils the composition law.

Let us further define the notion of complete positive divisibility of CP-divisibility of a dynamics.

Definition: **(Complete Positive Divisibility)** A quantum system undergoing time evolution characterized by trace preserving linear maps $\{\Lambda_{(t_2,t_1)}, t_2 \geq t_1 \geq t_0\}$ is complete positive divisible or CP-divisible if for every t_2 and t_1, $\Lambda_{(t_2,t_1)}$ is a completely positive map and fulfils the composition law.

Remark: We consider a dynamics of open quantum system as Markovian if it is complete positive divisible or CP-divisible. Consequently for non-Markovian dynamics complete positive divisibility breaks down.

At this point one may ask why the notion CP divisibility has been considered as the figure of merit. This is a legitimate question whose answer will be provided in the discussion.

As mentioned in the introduction an open quantum system is fragile and monotonically looses information to its environment. As a consequence of which system state distinguishability decreases along with time. If it is seen at certain point within the dynamics, that distinguishability is getting increased, this implies information is coming back from the environment to the system. BLP considered this backflow of information from environment to the system as the signature of non-Markovianity and defined non-Markovianity in terms of backflow of information. Before going to the main discussion let briefly illustrate some basic facts.

It is known that a norm always gives rise to the notion of distance. Distance between two quantum states inside the state space is defined in terms of trace norm. This distance is known as trace distance. For any two quantum states σ_1 and σ_2 their trace distance is defined as

$$D(\sigma_1, \sigma_2) = ||\sigma_1 - \sigma_2||_1$$

Physically this quantity gives the notion of ditinguishability between two quantum states. Suppose one party Alice prepares two quantum states σ_1 and σ_2 with equal probability and asks another party Bob to distinguish two states. Bob will perform a measurement and try to infer the states. It is true that not all the times Bob will guess the correct answer. Here the question of guessing probability comes into the picture. Bob will identify successfully which one of the two states has been given to him with the probability $\frac{1}{2}(1 + D(\sigma_1, \sigma_2))$. Clearly this shows trace distance can be linked with distinguishability of two states.

Let two initial states σ_1 and σ_2 be subjected to time evolution, then information flow is defined mathematically as

$$\mathcal{I} = \frac{d}{dt}||\Lambda_{(t,t_0)}\sigma_1 - \Lambda_{(t,t_0)}\sigma_2||_1$$

In BLP approach [113] Markovian dynamics is characterized by the condition $\mathcal{I} \leq 0$. Therefore according to BLP, for Markovian dynamics system monotonically looses information to its environment. On contrary, the condition $\mathcal{I} > 0$ implies information coming back from the environment to the system and hence distinguishability is getting increased. This is a signature of non-Markovianity. At this point one may ask the question what is the relationship between these two approaches. The following proposition serves the answer.

Proposition 3: [77] A dynamics which is Markovian in RHP sense, it is also Markovian in BLP sense.

Therefore we can say,

A dynamics is CP-divisible$\Longrightarrow$No information backflow from the environment to the system.

Therefore CP-divisibility breaking is necessary but not sufficient the have backflow of information form the environment.

This backflow of information causes the memory effect which is a resource for several information theoretic tasks. Hence we are interested to capture the resource theoretic essence of non-Markovian dynamics. Therefore we adopt CP-divisibility breaking of the dynamics as the figure of merit.

Next we shall illustrate another important idea regarding Markovian evolution. Differential equation plays an important role to understand any dynamics. Hence Markovian evolution can be studied in light of differential equation as well [77].

For a positive real number ϵ, the differential equation of the dynamics can be written as,

$$\frac{d}{dt}\sigma(t) = \lim_{\epsilon \to 0} \frac{[\Lambda_{(t+\epsilon,t)} - \mathbb{I}]}{\epsilon}\sigma(t) = \mathcal{L}(\sigma(t))$$

where the map $\mathcal{L} = \lim_{\epsilon \to 0} \frac{[\Lambda_{(t+\epsilon,t)} - \mathbb{I}]}{\epsilon}$ is called the generator of the evolution.

If the limit exists *i.e.* $\mathcal{L}$ is well defined then the dynamics is differentiable. The map will be given by

$$\Lambda(t+\epsilon, t) \equiv \mathbb{T} \exp\left(\int_t^{t+\epsilon} \mathcal{L} dt^{'}\right)$$

where $\mathbb{T}$ stands for the time ordering product. As a global consequence of the works by Kossakowski and his collaborators [118] and independently by Lindblad [117] there is a characterization of Markovian dynamics in terms of the form of the generator of the evolution. In the next theorem we shall state the characterization.

Theorem 5:(Kossakowski-Gorini-Sudarshan-Lindblad) [117,118] A map $\mathcal{L}_t$ is the generator of a quantum Markovian (or CP-divisible) dynamics if and only if it can be expressed in the form

$$\frac{d}{dt}\sigma(t) = \mathcal{L}_t(\sigma(t)) = -i[H(t), \sigma(t)] + \sum \gamma(t)[V_k(t)\sigma V_k^{\dagger}(t) - \frac{1}{2}\{V_k^{\dagger}(t)V_k(t), \sigma(t)\}]$$

where the operators $H(t)$ and $V_k(t)$ are time dependent and $H(t)$ is self adjoint and $\gamma(t) \geq 0$ for every k and t.

Here $V_k(t)$ s are known as Lindblad operators and the parameter values $\gamma(t)$s are called Lindblad coefficients. We shall restrict our discussions within the dynamics having such Lindblad type generators. Keeping these necessary prerequisites in mind next we shall discuss the upcoming chapters.

Chapter 3

Generating and detecting bound entanglement in two-qutrits using a family of indecomposable positive maps

Quantum entanglement is a magical gift of quantum theory that does not have any classical counterpart. It is clear from the chapter 1 and chapter 2, how nicely the theory of positive map is connected with the theory of bipartite entanglement. To understand the theory of entanglement, the theory of positive linear map needs to be explored deeply. The structure of positive linear maps is still uncharacterised even for most of the low dimensional algebra of operators. In this chapter we shall explore the the theory of positive maps on $\mathcal{B}(\mathbb{C}^3)$ further and discuss its connection with entanglement theory. This chapter is based on the work [122]. As mentioned in chapter 2 indecomposable positive linear maps are necessary to detect weak form of entanglement *i.e.* PPT entanglement. We shall introduce a novel one parameter family of positive maps on $\mathcal{B}(\mathbb{C}^3)$. This family of positive maps contains an indecomposable subfamily. Therefore it can be used to detect PPT entanglement. Our proposed family of map detects a class of PPT entangled state. We shall construct a family of entanglement witness out of this positive but non completely positive map. This class of entanglement witness is indecomposable in nature. We identify a class of entangled states which gets detected by this entanglement witness. We further show that the entanglement witness is weakly optimal. Moreover we construct a family of completely positive maps out of the proposed class of positive maps. This construction in turn gives rise to a two qutrit PPT entangled state. Construction of PPT entanglement was done earlier from indecomposable positive map in [26]. The approach was quite different. Facial structures of the convex cone of positive maps was harnessed in that construction. We have a taken a different approach to construct the PPT entangled state. Interestingly this new PPT entangled state can not be detected by some well known positive indecomposable maps.

3.1 One parameter family of indecomposable positive maps

Let us begin with the introduction to a new one parameter family of positive maps containing an indecomposable subfamily.

Definition: We define a one parameter class of linear trace preserving maps $\Lambda_\alpha : \mathcal{B}(\mathbb{C}^3) \to \mathcal{B}(\mathbb{C}^3)$ by,

$$\Lambda_\alpha(X) = \frac{1}{\alpha + \frac{1}{\alpha}} \begin{bmatrix} \alpha(x_{11} + x_{22}) & -x_{12} & -\alpha x_{13} \\ -x_{21} & \frac{x_{22}+x_{33}}{\alpha} & -x_{32} \\ -\alpha x_{31} & -x_{23} & \alpha x_{33} + \frac{x_{11}}{\alpha} \end{bmatrix} \tag{3.1}$$

where

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in \mathcal{B}(\mathbb{C}^3) \ \ \textit{and} \ \ \alpha \in (0, 1]. \tag{3.2}$$

Theorem 6: Λ_α is a positive map on $\mathcal{B}(\mathbb{C}^3)$ for all $0 < \alpha \leq 1$.

Proof: To prove the positivity of the linear map, it is sufficient to show that if acted upon any arbitrary pure state $|\phi\rangle = (\phi_1, \phi_2, \phi_3)^T$, the map will produce only positive semi-definite output. Here ϕ_1, ϕ_2, ϕ_3 are arbitrary complex numbers with the constraint $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = 1$. Let $\phi_1^*, \phi_2^*, \phi_3^*$ denote corresponding complex conjugates. Here we have

$$\Lambda\left(|\phi\rangle\langle\phi|\right) = \frac{1}{\alpha+\frac{1}{\alpha}}\begin{bmatrix} \alpha(|\phi_1|^2+|\phi_2|^2) & -\phi_1\phi_2^* & -\alpha\phi_1\phi_3^* \\ -\phi_1^*\phi_2 & \frac{|\phi_2|^2+|\phi_3|^2}{\alpha} & -\phi_2^*\phi_3 \\ -\alpha\phi_1^*\phi_3 & -\phi_2\phi_3^* & \alpha|\phi_3|^2+\frac{|\phi_1|^2}{\alpha} \end{bmatrix} \tag{3.3}$$

To prove the positivity of the matrix $\Lambda_\alpha\left(|\phi\rangle\langle\phi|\right)$, we need to show that all of its principal minors are positive. The 1st order principal minors are the diagonal elements, which are positive for any $\alpha > 0$. The three 2nd order principal minors are

$$M_1 = \left(\frac{1}{\alpha+\frac{1}{\alpha}}\right)^2 \begin{vmatrix} \alpha(|\phi_1|^2+|\phi_2|^2) & -\phi_1\phi_2^* \\ -\phi_1^*\phi_2 & \frac{|\phi_2|^2+|\phi_3|^2}{\alpha} \end{vmatrix},$$

$$M_2 = \left(\frac{1}{\alpha+\frac{1}{\alpha}}\right)^2 \begin{vmatrix} \alpha(|\phi_1|^2+|\phi_2|^2) & -\alpha\phi_1\phi_3^* \\ -\alpha\phi_1^*\phi_3 & \alpha|\phi_3|^2+\frac{|\phi_1|^2}{\alpha} \end{vmatrix}, \tag{3.4}$$

$$M_3 = \left(\frac{1}{\alpha+\frac{1}{\alpha}}\right)^2 \begin{vmatrix} \frac{|\phi_2|^2+|\phi_3|^2}{\alpha} & -\phi_2^*\phi_3 \\ -\phi_2\phi_3^* & \alpha|\phi_3|^2+\frac{|\phi_1|^2}{\alpha} \end{vmatrix}.$$

We perform the simplification and find that

$$M_1 = \frac{\alpha^2}{(1+\alpha^2)^2}\left(|\phi_2|^4 + |\phi_1|^2|\phi_3|^2 + |\phi_2|^2|\phi_3|^2\right)$$

. Therefore M_1 is non negative as $\alpha \in (0,1]$. Similarly, we find

$$M_2 = \frac{\alpha^2}{(1+\alpha^2)^2}\left(|\phi_1|^4 + |\phi_1|^2|\phi_2|^2 + |\phi_2|^2|\phi_3|^2\alpha^2\right)$$

which is also a non negative quantity, and

$$M_3 = \frac{\alpha^2}{(1+\alpha^2)^2}\left(|\phi_3|^4 + \frac{1}{\alpha^2}|\phi_1|^2|\phi_2|^2 + \frac{1}{\alpha^2}|\phi_1|^2|\phi_3|^2\right)$$

which is again a non negative quantity as $\alpha \in (0, 1]$.

The remaining principal minor is the determinant of the matrix $\Lambda_\alpha (|\phi\rangle \langle\phi|)$,

The determinant is given by

$$D = \frac{\alpha^4}{(1+\alpha^2)^3}\left[|\phi_2|^2|\phi_3|^4 + \frac{|\phi_3|^2|\phi_1|^4}{\alpha^2} + \frac{|\phi_1|^2|\phi_2|^4}{\alpha^2}\right] - \frac{\alpha^4}{(1+\alpha^2)^3}\left[2|\phi_1|^2|\phi_2|^2|\phi_3|^2 + 2|\phi_1|^2 Re(\phi_2^*\phi_3)^2) - \frac{1}{\alpha^2}|\phi_1|^2|\phi_2|^2|\phi_3|^2\right],$$

Since $\mathrm{Re}(\phi_2^*\phi_3)^2 \leq |\phi_2|^2|\phi_3|^2, \forall\ \phi_2$ and ϕ_3, we have

$$\begin{aligned} D &\geq \frac{\alpha^4}{(1+\alpha^2)^3}\left[|\phi_2|^2|\phi_3|^4 + \frac{|\phi_3|^2|\phi_1|^4}{\alpha^2} + \frac{|\phi_1|^2|\phi_2|^4}{\alpha^2} - (4 - \frac{1}{\alpha^2})|\phi_1|^2|\phi_2|^2|\phi_3|^2\right] \\ &\geq \frac{\alpha^4}{(1+\alpha^2)^3}\left[|\phi_2|^2|\phi_3|^4 + |\phi_3|^2|\phi_1|^4 + |\phi_1|^2|\phi_2|^4 - 3|\phi_1|^2|\phi_2|^2|\phi_3|^2\right], \end{aligned}$$

for all $\alpha \leq 1$. Here $\mathrm{Re}(\cdot)$ means the real part of a complex number. It is straightforward to check that the quantity

$$\left[|\phi_2|^2|\phi_3|^4 + |\phi_3|^2|\phi_1|^4 + |\phi_1|^2|\phi_2|^4 - 3|\phi_1|^2|\phi_2|^2|\phi_3|^2\right] \geq 0,$$

for all ϕ_1, ϕ_2, ϕ_3 with the constraint $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = 1$. Therefore, the map $\Lambda_\alpha(\cdot)$ is positive for all $0 < \alpha \leq 1$. □

Our aim is to find whether the map Λ_α can detect entangled states which are positive under partial transposition. For this purpose, we prove the following corollary.

Corollary 1: Λ_α is a non completely positive indecomposable map for each $\alpha \in (0, 1]$.

Proof. At first we have to show that the given positive map is not completely positive. For this purpose, using CJ isomorphism, it is sufficient to show that $\mathbb{I} \otimes \Lambda_\alpha(|\Phi\rangle \langle\Phi|)$ is not positive semi-definite. Here, $|\Phi\rangle$ is the maximally entangled two qutrit state.

Let us consider the corresponding Choi matrix first. We take the maximally entangled state for two qutrit system as $|\Phi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right)$ where,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.5}$$

The one sided action of the map on the maximally entangled state produces the Choi matrix,

$$\mathcal{C}^{\Lambda_\alpha} = \begin{bmatrix} \frac{\alpha^2}{3+3\alpha^2} & 0 & 0 & 0 & -\frac{\alpha}{3+3\alpha^2} & 0 & 0 & 0 & -\frac{\alpha^2}{3+3\alpha^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3+3\alpha^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha^2}{3+3\alpha^2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\alpha}{3+3\alpha^2} & 0 & 0 & 0 & \frac{1}{3+3\alpha^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\alpha}{3+3\alpha^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\alpha}{3+3\alpha^2} & 0 & \frac{1}{3+3\alpha^2} & 0 \\ -\frac{\alpha^2}{3+3\alpha^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha^2}{3+3\alpha^2} \end{bmatrix}. \tag{3.6}$$

The least eigenvalue of $\mathcal{C}^{\Lambda_\alpha}$ is $\lambda' = \frac{1-\sqrt{1+4\alpha^2}}{6+6\alpha^2}$. We note that it is a negative quantity within the above parameter range $\alpha \in (0,1]$. Hence, it shows that the given map is not completely positive.

To prove the indecomposability of the map, we have to show that it can detect at least one entangled state which is positive under partial transposition. Such a class of two qutrit entangled states [31] is the following

$$\tau_x = \frac{1}{3(1+x+x^{-1})} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.7}$$

with x being any positive real number. Applying the proposed map, we have

$$
\mathbb{I} \otimes \Lambda_\alpha(\tau_x) = \frac{N}{3(1+x+x^{-1})} \begin{pmatrix} \alpha(x+1) & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -\alpha \\ 0 & \frac{x+\frac{1}{x}}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{x}+\frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\left(1+\frac{1}{x}\right) & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \frac{x+1}{\alpha} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x\alpha+\frac{1}{\alpha x} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha\left(x+\frac{1}{x}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{1+\frac{1}{x}}{\alpha} & 0 \\ -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x}{\alpha}+\alpha \end{pmatrix}
$$

Here, $N = 1/(\alpha + 1/\alpha)$ is the normalization factor. One of the principal minors of the matrix $\mathbb{I} \otimes \Lambda_\alpha(\tau_x)$ is given by

$$
\begin{aligned}
D_{\tau_x} = N & \begin{vmatrix} \alpha(1+x) & -1 & -\alpha \\ -1 & \frac{1+x}{\alpha} & 0 \\ -\alpha & 0 & \alpha + \frac{x}{\alpha} \end{vmatrix} \\
&= N[x(2+x)(\alpha + \tfrac{x}{\alpha}) - \alpha(1+x)].
\end{aligned}
$$

It is clear that for any $\alpha \in (0, 1]$ the function $f(x) = x(2+x)(\alpha + \frac{x}{\alpha}) - \alpha(1+x)$ is a continuous function and $\lim_{x \longrightarrow 0} f(x) = -\alpha < 0$. So $D_{\tau_x} < 0$ for some small $x > 0$ and hence Λ_α is indecomposable. $\square$

The negativity of D_{τ_x} for a large range of parameters also can be seen from Fig. 3.1 where D_{τ_x} is plotted with respect to x for some particular values of α. The following cases may be considered a examples.

Case 1: Considering $\alpha = \frac{1}{4}$, we see that $D_{\tau_x} < 0$, if $x < 0.154$.

Case 2: Considering $\alpha = \frac{1}{2}$, we see that $D_{\tau_x} < 0$, if $x < 0.269$.

Case 3: Let us now consider $\alpha = 1$. In this case we can see that $D_{\tau_x} < 0$, if $x < \sqrt{2} - 1$.

Therefore, the one parameter class of maps contains indecomposable positive maps.

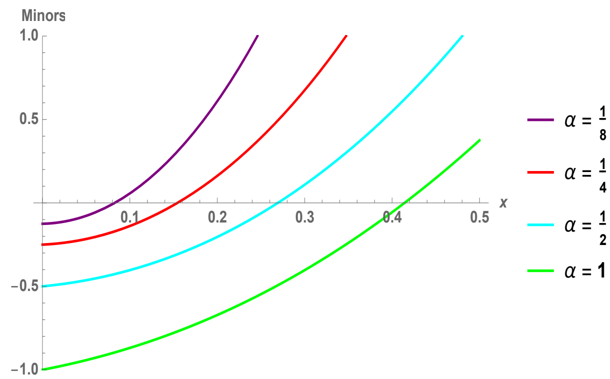

Figure 3.1: (Colour online) Variation of Minors D_{τ_x} with respect to map parameter α

In the following corollary let us now illustrate the dual map corresponding to the positive map introduced in the definition 1.

Corollary 2: The following map

$$\Lambda_{\alpha}^{\dagger}(X) = \frac{1}{\alpha + \frac{1}{\alpha}} \begin{bmatrix} \alpha(x_{11} + x_{33}) & -x_{12} & -\alpha x_{13} \\ -x_{21} & \frac{x_{22}+x_{11}}{\alpha} & -x_{32} \\ -\alpha x_{31} & -x_{23} & \alpha x_{33} + \frac{x_{22}}{\alpha} \end{bmatrix} \tag{3.8}$$

is also positive and indecomposable in the range $0 < \alpha \leq 1$.

Proof. The proof of positivity follows similarly to that of **Theorem 6** (3.1). For the proof of indecomposability, we construct the matrix $\mathbb{I} \otimes \Lambda_{\alpha}^{\dagger}(\tau_x)$, to find that it will have at least one negative eigenvalue for (irrespective of the value of α)

$$x > \frac{1}{\sqrt{2} - 1}.$$

□

3.2 Entanglement witness and Weak optimality

Our next goal is to construct a class of entanglement witness [12]. This will set the experimental viability of our findings on a firm footing. Moreover, we also prove that at least one of our constructed witnesses is weakly optimal.

As mentioned earlier any positive but not completely positive map gives rise to an entanglement witness. For a given map $\varGamma$, its corresponding Choi matrix $\mathcal{C}^{\varGamma}$ serves as a witness for some entangled state. An entanglement witness $\mathcal{W}$ is said to be weakly optimal [123] if there exists some pure product state $|\gamma\rangle \otimes |\delta\rangle$ such that

$$\langle\gamma| \otimes \langle\delta| \mathcal{W} |\gamma\rangle \otimes |\delta\rangle = 0$$

.

It is known that for two positive semi-definite matrices A and B, the inequality $\mathrm{Tr}[AB] \geq 0$ is always true. Based on this fact, we have $\mathrm{Tr}[|\varPhi\rangle\langle\varPhi| \mathbb{I} \otimes \varLambda_{\alpha}^{\dagger}(\sigma)] \geq 0$, for all separable states σ and for at least one entangled state, the trace inequality will acquire negative value. Following the trace rule $\mathrm{Tr}[CD] = \mathrm{Tr}[DC]$ for any pair of matrices C and D, we get

$$\mathrm{Tr}[|\varPhi\rangle\langle\varPhi| \mathbb{I} \otimes \varLambda_{\alpha}^{\dagger}(\rho)] = \mathrm{Tr}[\mathbb{I} \otimes \varLambda_{\alpha}(|\varPhi\rangle\langle\varPhi|)\rho] \geq 0,$$

for any separable state ρ of 3×3 dimension. This can be extended to arbitrary $d \times 3$ dimensional systems.

We can thus consider the one parameter family of positive but not completely positive maps $\varLambda_{\alpha}$ with the corresponding Choi matrix $\mathcal{C}^{\varLambda_{\alpha}} = \mathbb{I} \otimes \varLambda_{\alpha}(|\varPhi\rangle\langle\varPhi|)$, to be an one parameter family of entanglement witnesses. Now, for $\alpha = 1$, we find $|\gamma\rangle = |\delta\rangle = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$ such that

$$\langle\gamma| \otimes \langle\gamma| \mathcal{C}^{\varLambda_{\alpha}} |\gamma\rangle \otimes |\gamma\rangle = 0.$$

As entanglement witness is a Hermitian operator, it is basically an observable. There is a local implementation of this weakly optimal witness. The witness can be expressed as a linear sum of two qutrit local observables. We consider the 3×3

identity matrix $G_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, along with 8 Gell-Mann matrices

$$G_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, G_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, G_6 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, G_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$G_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, G_9 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

as 9 local observables as they are Hermitian. We denote them as $G_i, i = 1.....9$. We note that for $\alpha = 1$,

$$\begin{aligned} \mathcal{C}^{\Lambda_1} = {} & \tfrac{1}{3} G_1 \otimes G_1 - \tfrac{1}{6} G_2 \otimes G_2 + \tfrac{1}{6} G_3 \otimes G_3 + \tfrac{1}{12} G_4 \otimes G_4 - \tfrac{1}{4\sqrt{3}} G_4 \otimes G_9 \\ & - \tfrac{1}{6} G_5 \otimes G_5 + \tfrac{1}{6} G_6 \otimes G_6 - \tfrac{1}{6} G_7 \otimes G_7 - \tfrac{1}{6} G_8 \otimes G_8 \\ & + \tfrac{1}{4\sqrt{3}} G_9 \otimes G_4 + \tfrac{1}{12} G_9 \otimes G_9. \end{aligned}$$

This witness is of course indecomposable as the corresponding positive map is indecomposable. To further establish this fact, we apply the witness $\mathcal{C}_{\Lambda_1}$ on τ_x (3.7) to find

$$\mathrm{Tr}\left[\mathcal{C}^{\Lambda_1} \tau_x\right] = \frac{3 - x}{18\left(x^2 + x + 1\right)}. \tag{3.9}$$

Thus the witness detects entanglement of the two-qutrit PPT entangled state τ_x for $x > 3$.

3.3 Structural physical approximation and construction of a two qutrit bound entangled state

In general positive linear maps have a physical limitation. A positive map can not be realised in the laboratory. Hence in general a positive map is unphysical. Within the class of positive maps, completely positive maps can be implemented physically. Construction of a completely positive map corresponding to a given positive map is known as structural physical approximation (SPA). Mathematically this idea has a nice geometric essence. As mentioned earlier the set of positive linear maps is a convex set. Given a positive linear map if a line is drawn to the completely depolarizing map from it, the points of intersection of this line with the set of completely positive maps provide the structural physical approximations of the map. Obviously the question of optimal SPA is relevant in this context. A protocol to find the optimal SPA corresponding to a given positive map was first given in [124]. We formulate the SPA corresponding to our one parameter family of maps Λ_α and establish that it produces a PPT entangled states.

We considered the Choi matrix $\mathcal{C}^{\Lambda_\alpha}$ of the family of maps and found the least eigenvalue of $\mathcal{C}^{\Lambda_\alpha}$ to be $\lambda^{'} = \frac{1-\sqrt{1+4\alpha^2}}{6+6\alpha^2}$ when we take $\alpha \in (0, 1]$. It is a negative quantity within the above parameter range. Defining $\lambda = max\left[0, -\lambda^{'}\right]$, and following the prescription of [124], the optimal SPA map corresponding to the map Λ_α is given by

$$\Lambda_\alpha^{opt} = p^0 \Lambda_{dep} + (1 - p^0)\Lambda_\alpha^{'}$$

where $p^0 = \frac{\lambda d d^{'} \beta_{\Lambda_\alpha}^{-1}}{\lambda d d^{'} \beta_{\Lambda}^{-1} + 1}$, $\Lambda_{dep} = \frac{Tr(.)}{d^{'}}\mathbb{I}$ is the completely depolarizing map and $\Lambda^{'} = \beta_\Lambda^{-1}\Lambda$ is the re-scaling of the original map. Here $d = d^{'} = 3$, the input and output dimension of the map Λ_α, and as a consequence of trace preservation, the value of $\beta_{\Lambda_\alpha} = 1$. Therefore, the optimal SPA map $\Lambda_\alpha^{opt} : \mathcal{B}(\mathbb{C}^3) \rightarrow \mathcal{B}(\mathbb{C}^3)$ is given by

$$\Lambda_{\alpha}^{opt}\left(X\right) =$$

$$\begin{bmatrix} \frac{x_{33}(m-1)+(x_{11}+x_{22})\left(2\alpha^2+m-1\right)}{2\alpha^2+3m-1} & \frac{2x_{12}\alpha}{-2\alpha^2-3m+1} & \frac{2x_{13}\alpha^2}{-2\alpha^2-3m+1} \\ \frac{2x_{21}\alpha}{-2\alpha^2-3m+1} & \frac{x_{11}(m-1)+(x_{22}+x_{33})(m+1)}{2\alpha^2+3m-1} & \frac{2x_{32}\alpha}{-2\alpha^2-3m+1} \\ \frac{2x_{31}\alpha^2}{-2\alpha^2-3m+1} & \frac{2x_{23}\alpha}{-2\alpha^2-3m+1} & \frac{-x_{22}+x_{33}\left(2\alpha^2-1\right)+x_{11}(m+1)+(x_{22}+x_{33})m}{2\alpha^2+3m-1} \end{bmatrix} \tag{3.10}$$

where $m = \sqrt{4\alpha^2+1}$. We note that the SPA map is also trace preserving. To check the completely positivity of the SPA map, we compute the corresponding Choi matrix.

The Choi matrix is $\quad \mathcal{C}^{\Lambda_{\alpha}^{opt}} =$

$$\begin{bmatrix} \frac{2\alpha^2+m-1}{R} & 0 & 0 & 0 & \frac{2\alpha}{-R} & 0 & 0 & 0 & \frac{2\alpha^2}{-R} \\ 0 & \frac{m-1}{R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m+1}{R} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\alpha^2+m-1}{R} & 0 & 0 & 0 & 0 & 0 \\ \frac{2\alpha}{-R} & 0 & 0 & 0 & \frac{m+1}{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{m-1}{R} & 0 & \frac{2\alpha}{-R} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{m-1}{R} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2\alpha}{-R} & 0 & \frac{m+1}{R} & 0 \\ \frac{2\alpha^2}{-R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\alpha^2+m-1}{R} \end{bmatrix}. \tag{3.11}$$

Where $R = 6\alpha^2 + 9m - 3$ and $m = \sqrt{4\alpha^2+1}$.

We next compute the eigenvalues of the 9×9 Choi matrix $\mathcal{C}^{\Lambda_{\alpha}^{opt}}$ and observe that the Choi matrix is positive semi-definite for the whole range of α. This signifies the complete positivity of the SPA map. Moreover, it is to be noted that the Choi matrix is a valid density matrix as $\mathcal{C}^{\Lambda_{\alpha}^{opt}}$ is Hermitian, positive semi-definite and of trace 1 for $\alpha \in (0,1]$. Hence, we obtain a one parameter family of two qutrit states.

The partial transposition of $\mathcal{C}^{\Lambda_{\alpha}^{opt}}$ is given by

$$\left(\mathcal{C}^{\Lambda_{\alpha}^{opt}}\right)^{T} =$$

$$\begin{pmatrix} \frac{2\alpha^2+m-1}{R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{m-1}{R} & 0 & \frac{2\alpha}{-R} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m+1}{R} & 0 & 0 & 0 & \frac{2\alpha^2}{-R} & 0 & 0 \\ 0 & \frac{2\alpha}{-R} & 0 & \frac{2\alpha^2+m-1}{R} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{m+1}{R} & 0 & 0 & 0 & \frac{2\alpha}{-R} \\ 0 & 0 & 0 & 0 & 0 & \frac{m-1}{R} & 0 & 0 & 0 \\ 0 & 0 & \frac{2\alpha^2}{-R} & 0 & 0 & 0 & \frac{m-1}{R} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{m+1}{R} & 0 \\ 0 & 0 & 0 & 0 & \frac{2\alpha}{-R} & 0 & 0 & 0 & \frac{2\alpha^2+m-1}{R} \end{pmatrix}.$$

We compute its eigenvalues and plot them with respect to α in Figure 3.2. We note that among the nine eigenvalues, one eigenvalue is negative in the interval $\alpha \in (0,1)$, and other eight eigenvalues are all positive for $\alpha \in (0,1)$. Hence, the class of states for this interval of values of α is NPPT, and therefore, it is entangled. Interestingly, for the parameter value $\alpha = 1$, all the eigenvalues of the partially transposed matrix are non negative, and hence, the state $\mathcal{C}^{\Lambda_\alpha^{opt}}$ is PPT for $\alpha = 1$.

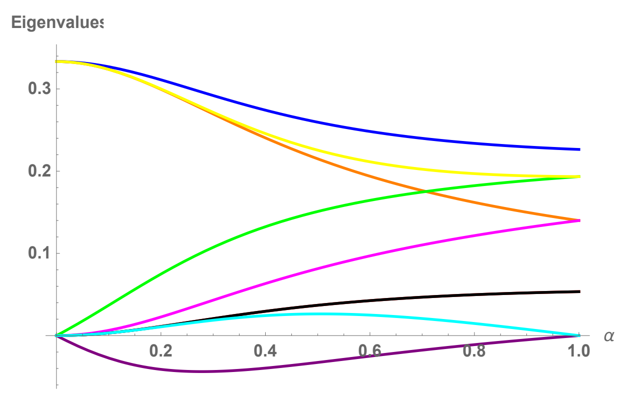

Figure 3.2: (Colour online) The eigenvalues of the partially transposed matrix $\left(\mathcal{C}^{\Lambda_\alpha^{opt}}\right)^T$ are plotted with respect to α. There are eight eigenvalues in the plot since two of the nine eigenvalues are equal in $(0,1]$

We now try to find whether the state $\mathcal{C}^{\Lambda_1^{opt}}$ is entangled. Since we are dealing with a two-qutrit system, the partial transposition criterion is no longer useful for entanglement detection. We adopt the covariance matrix criterion [125, 126] for entanglement detection. We can consider the 3 by 3 identity matrix along with the 8 Gell-Mann matrices as orthogonal local observables as they are orthogonal and Hermitian. We take the Choi state $\mathcal{C}^{\Lambda_\alpha^{opt}}$ and obtain its two reduced density matrices $\mathcal{C}_A^{\Lambda_\alpha^{opt}}$ and $\mathcal{C}_B^{\Lambda_\alpha^{opt}}$ respectively. According to the covariance matrix criterion [125, 126], for separable states,

$$||C||_1 \leq \sqrt{\left(1 - \mathrm{Tr}\left[\left(\mathcal{C}_A^{\Lambda_\alpha^{opt}}\right)^2\right]\right)\left(1 - \mathrm{Tr}\left[\left(\mathcal{C}_B^{\Lambda_\alpha^{opt}}\right)^2\right]\right)} \tag{3.12}$$

where $||.||_1$ stands for the trace norm and the components of the C matrix are given by

$$C_{ij} = \langle H_i^A \otimes H_j^B \rangle - \langle H_i^A \rangle \langle H_j^B \rangle \tag{3.13}$$

and H_i^A and H_j^B denote local orthogonal observables on two sides. We evaluate the C matrix using the state $\mathcal{C}^{\Lambda_\alpha^{opt}}$ and its reduced density matrices, and obtain that the LHS of Eq.(3.12) is strictly greater than the RHS for the values of α in $(0, 1]$. The result has been illustrated in 3.3 This certifies the presence of entanglement in the state $\mathcal{C}^{\Lambda_\alpha^{opt}}$ for $\alpha \in (0, 1]$. Therefore, the state corresponding to the value of the parameter $\alpha = 1$, *i.e.*, $\mathcal{C}^{\Lambda_1^{opt}}$ is PPT-entangled. So, the SPA map Λ_α^{opt} can generate a two-qutrit PPT entangled state $\mathcal{C}^{\Lambda_1^{opt}}$.

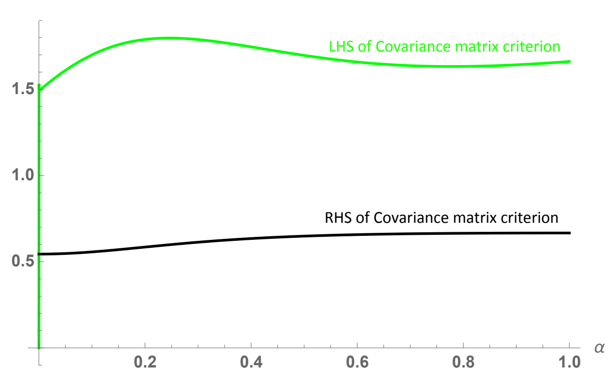

Figure 3.3: (Colour online) LHS and RHS of Eq.(3.12) are plotted versus α.

The state $\mathcal{C}^{\Lambda_\alpha^{opt}}$ is NPPT for the values of the parameter α in $(0,1)$ and therefore, it is entangled.

Finally, we check whether the PPT entangled state $\mathcal{C}^{\Lambda_1^{opt}}$ can be detected by some other existing positive maps. PPT entangled states are considered as a weak form of entanglement that is usually very hard to detect. As discussed earlier, indecomposable maps are necessary to detect PPT entangled states. The celebrated Choi map $\phi_{\text{choi}} : \mathcal{B}(\mathbb{C}^3) \to \mathcal{B}(\mathbb{C}^3)$ [8], one of the first examples of indecomposable maps in the literature, is defined as

$$\phi_{\text{Choi}}(X) = \begin{bmatrix} x_{11}+x_{22} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{22} \end{bmatrix} \tag{3.14}$$

where

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in \mathcal{B}(\mathbb{C}^3).$$

A positive map Λ is said to detect an entangled state κ if and only if $\mathbb{I}\otimes\Lambda(\kappa) \ngeqslant \Theta$, where Θ stands for zero operator. It can be checked that $\mathbb{I}\otimes\phi_{\text{Choi}}\left(\mathcal{C}_{\Lambda_\alpha^{opt}}\right) \geqslant \Theta \ \forall\alpha \in (0,1]$. Hence, the Choi map is unable to detect the above PPT entangled state.

Recently, Miller and Olkiewicz [59] introduced another indecomposable map ϕ_{MO} on $\mathcal{B}(\mathbb{C}^3)$ given by,

$$\phi_{\text{MO}}(X) = \begin{bmatrix} \frac{1}{2}(x_{11}+x_{22}) & 0 & \frac{1}{\sqrt{2}}x_{13} \\ 0 & \frac{1}{2}(x_{11}+x_{22}) & \frac{1}{\sqrt{2}}x_{32} \\ \frac{1}{\sqrt{2}}x_{31} & \frac{1}{\sqrt{2}}x_{23} & x_{33} \end{bmatrix} \tag{3.15}$$

where

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in \mathcal{B}(\mathbb{C}^3).$$

It can be again checked that $\mathbb{I} \otimes \phi_{\mathrm{MO}} \left(\mathcal{C}^{\Lambda_\alpha^{opt}}\right) \geqslant \Theta \quad \forall \alpha \in (0,1]$. Hence the above map also cannot detect the PPT entangled state $\mathcal{C}^{\Lambda_1^{opt}}$.

To summarize the chapter, a novel positive indecomposable maps has been constructed on the space of operators acting on three dimensional complex Hilbert space. This map gives rise to an indecomposable entanglement witness which can detect a class of PPT entangled state. The entanglement witness is weakly optimal and can be decomposed in terms of tensor product of qutrit observables. The structural physical approximation of the positive map has been constructed which in turn gives rise two a class of two qutrit entangled state.Within this class, for a particular value of the parameter, PPT entanglement is observed.

Next we shall focus on the dynamics of open quantum system. We shall develop one by one, the resource theoretic ideas non-Markovianity, discuss resource detection and resource interconversion. Let us move to the details of these ideas in the following chapters.

Chapter 4

Convex resource theory of non-Markovianity

Resource theory provides a formal platform to study various resources [127] necessary to perform different tasks. The construction of resource theories in connection with various quantum phenomena such as entanglement [11, 128], coherence [129], reference frame and asymmetry [130], nonlocality [131], non-gaussianity [132], and thermodynamics [133, 134], has flourished in recent years. Resource theory can be broadly classified into two categories : static and dynamic [127]. Resource theories of entanglement and coherence fall under the first category. Recently the idea of dynamical resource theories has evolved in a series of works [135–138]. Since dynamics of a quantum system is governed by completely positive trace preserving (CPTP) maps, to identify a dynamical resource one needs to characterize CPTP maps which is unable to generate the corresponding resource. Famous CJ isomorphism can be a useful tool here. As mentioned in the introduction non-Markovianity in terms of backflow of information is a resource in various tasks. Non-Markovianity being a property of a dynamics, can also be regarded as dynamical resource. However, non-Markovianity is a property of time parametrized family of CPTP maps. Therefore in this context one needs to consider maps depending on time. In other dynamical resource theories, resourceless operations do not change in time. But in the case of non-Markovian dynamics the resource can be generated anywhere within the dynamics, and hence, the construction of resource theory of non-Markovianity is different from other resource theories.

Previously, there has been some attempts to construct resource theory of non-Markovianity based on various approaches [119, 139, 140]. A formalism based on a tripartite scenario was employed in [139] where the classical notion of non-Markovianity emerges as a special case of the approach. It may be noted that the phenomenon of non-Markovianity is not restricted to the domain of quantum theory [141]. In another resource theoretic approach to address non-Markovianity, genuine quantum memory has been proposed as a resource in [140] where a game played between two players to certify quantum memory has been devised. Further, a resource theory of non-Markovianity has been formulated recently via the process tensor formalism [119], where the concept of superprocesses has been defined which transforms the processes and characterizes multi-time processes in a client-server scenario. Additionally, non-Markovianity monotones have also been proposed in terms of quantum mutual information and quantum coherence [142,143], though without a formal resource theoretic framework.

In the present chapter which is based on the work [144] we employ the CJ isomorphism to construct a convex resource theory of non-Markovianity by identifying free and resourceful operations. Our resource theory of non-Markovianity encapsulates information backflow caused by CP-divisibility breaking of quantum channels as the central feature. Note that, though divisible operations do not exhaust all Markovian operations [145, 146], CP-divisible non-Markovian operations cannot generate information backflow. Hence, it is legitimate to consider only indivisible operations as resourceful operations. Our resource theory of non-Markovianity is valid for arbitrary finite dimensional single or *any*-partite systems, satisfying all the basic ingredients of a resource theory [147]. In our resource theoretic framework, we define the robustness of non-Markovianity and relate it to the RHP measure [80]. Further, robustness defined by us, has an explicit physical implication on the quantum capacity of dephasing channels.

Our approach is thus quite different from the above-mentioned resource theories of non-Markovianity [119, 139, 140]. Our formalism closely parallels other information theoretic resource theories such as of entanglement in the sense of providing a way to define the concept of robustness of non-Markovianity, as well as to construct a theory of non-Markovianity witnesses [148, 149]. A clear difference between our approach and the resource theories of entanglement though lies on the significance of free and resourceful operations, as explained in details in the next section.

4.1 Resource theory of non-Markovianity

Several physical phenomena in quantum mechanics are defined directly at the level of quantum states. In those cases constraints are imposed over the physical operations. If one wishes to construct a resource theory concerning one of such phenomena, identification of the states containing the signature of such resources is essential; or conversely, one needs to identify the resourceless states, defined as the "free states". One of the important components of quantum resource theory is the set of restricted quantum operations under which the resource cannot be generated. These operations are called the "free operations". In case of non-Markovianity, it should be kept in mind that non-Markovianity is a property of quantum processes, rather than that of states. So such a resource is basically a dynamic in nature. Hence, in this respect the resource theory of non-Markovianity is fundamentally different from the existing resource theories of entanglement, coherence or thermodynamics. Here, we do not need to identify the free states.

On contrary, identification of the resourceless operations is important in the resource theory of non-Markovianity. Automatically, these would identify the resourceful operations. From now onwards the term "free operations" will be used to imply resourceless operations in the context of non-Markovianity. (Here it is important not to get confused with the term "free operations" usually used in the resource theory of entanglement or thermodynamics). Further, we need to quantify the resource and hence we need a resource quantifier. For this purpose convex structure of the resource theory adds technical convenience. Quantum resource theories such as entanglement, coherence, asymmetry and athermality possess such structure. We construct these basic components for the resource theory of non-Markovianity and establish that it satisfies all the requirements of a convex resource theory under a particular constraint.

Free operations: We define the complete positive divisible operations as the resourceless or free operations. They are expressed as

$$\Lambda^{\mathcal{M}}(t_2, t_1) \equiv \mathbb{T} \exp\left(\int_{t_1}^{t_2} \mathcal{L}_t dt\right), \tag{4.1}$$

where $\mathcal{L}_t(\cdot) = \sum_k \Gamma_k(t)\left(L_k(\cdot)L_k^\dagger - \{L_k^\dagger L_k, (\cdot)\}\right)$ with $\Gamma_k(t) \geq 0, \forall k, t$ and $\mathbb{T}$ stands for the time ordering product. It is important to mention here that in general complete positive divisible operations do not form a convex set [150, 151]. To have a convex resource theory we put a further constraint on the definition of resourceless operations, given by the small time interval approximation.

To deal with the operations, we have to work with positive maps acting on the space of bounded operators on some Hilbert space. Choi-Jamilkowski isomorphism [6, 7] plays an important role in this context. As the CJ isomorphism (2.1) translates everything to the language of operators from the language of maps, we shall use the Choi operator corresponding to a dynamical map for the detailed study. Considering the dynamical evolution between time intervals t_1 and t_2 where $t_2 > t_1$, and denoting t_2 as $t + \epsilon$ and t_1 as t for some positive ϵ , we can define

$$\mathcal{C}^{\mathcal{S}}(t+\epsilon, t) = \mathbb{I} \otimes \Lambda^{\mathcal{S}}(t+\epsilon, t)(|\psi\rangle\langle\psi|) \quad with \quad \epsilon > 0, \tag{4.2}$$

where $|\psi\rangle$ is the maximally entangled state of $d \times d$ dimension, for a d dimensional system. We note that for a divisible operation, the Choi operator corresponding to the the map is a valid density matrix and hence a quantum state.

Well known Rivas-Huelga-Plenio (RHP) measure of non-Markovianity [80] is given by

$$g^{\mathcal{N}}(t) = \lim_{\epsilon \to 0^+} \frac{\parallel \mathcal{C}^{\mathcal{N}}(t+\epsilon, t) \parallel_1 - 1}{\epsilon}, \tag{4.3}$$

where $\parallel \cdot \parallel_1$ is the trace norm and $\mathcal{C}^{\mathcal{N}}(t+\epsilon, t) = \mathbb{I} \otimes \Lambda^{\mathcal{N}}(t+\epsilon, t)(|\psi\rangle\langle\psi|)$ is the Choi operator corresponding to any operation $\Lambda_{\mathcal{N}}$. For a divisible evolution $\Lambda_{\mathcal{M}}$, $g^{\mathcal{M}}(t)$ is always zero since $\mathcal{C}^{\mathcal{M}}(t+\epsilon, t)$ is a valid state and $\parallel \mathcal{C}^{\mathcal{M}}(t+\epsilon, t) \parallel_1 = 1$. Restricting ourselves within finite dimensional systems, we have considered implicitly the trace of Choi operator corresponding to a free operation to be unity. This is because of the fact that the operation is always trace preserving. Therefore, for divisible operations, the output is always a valid state. So both the trace and trace norm of such outputs will always be unity. On the other hand, indivisible operations will give rise to negative eigenvalues of the output. For those cases the trace and Hermiticity preserving features of the map ensure that the trace norm of the output will always exceed unity.

We only consider those operations having Lindblad type generators as $\Lambda(t+\epsilon, t) \equiv \mathbb{T} \exp\left(\int_t^{t+\epsilon} \mathcal{L} dt'\right)$. Moreover, if we consider

$$\epsilon \Gamma_k(t) << 1 \quad (\forall k), \tag{4.4}$$

which we term as the 'small time interval approximation', we can take the operation as $\Lambda(t+\epsilon, t) \equiv \mathbb{I} + \epsilon \mathcal{L}$, by considering only upto 1st order terms in the Taylor series expansion. Thus we collect Choi operators corresponding to all such families of dynamical maps and define the set of free Choi operators as

$$\mathbb{F}^{\epsilon}_{\mathcal{M}} = \left\{ \mathcal{C}^{\mathcal{M}}_{\alpha}(t+\epsilon, t) \quad : \alpha \in J \ \ and \ \ \parallel \mathcal{C}^{\mathcal{M}}_{\alpha}(t+\epsilon, t) \parallel_1 = 1 \ \forall \ t, \epsilon \ \alpha \right\}, \tag{4.5}$$

with the condition $\epsilon \Gamma_k(t) << 1 \quad (\forall k)$. Here J is any index set. In general the Lindblad operators are non commutative at different times, requiring a time ordering operation while defining the set $\mathbb{F}^{\epsilon}_{\mathcal{M}}$. However, in our present small time interval approximation, no time ordering is required since there is no higher order terms.

We now prove the following propositions based on previous definition. These establish particularly the essential properties of the resource theory. Before proceeding to the technical proofs one should keep in mind again that the properties discussed here are the features of resourceless dynamical maps, *viz.*, divisible maps. Exploiting CJ isomorphism [6, 7], we shall prove the propositions at the level of corresponding Choi operators. Note that we will be defining a resource which is the sole property of the operation, not the state. Therefore, in the physical picture, there is no concept of free or resourceful states in this context.

Proposition 4: In the limit of $\epsilon \mathit{\Gamma}_k(t) << 1 \ \ (\forall k)$, the set of free Choi operators form a convex set.

Proof. Let $\mathcal{C}^{\mathcal{M}}_{(1)}(t+\epsilon, t)$ and $\mathcal{C}^{\mathcal{M}}_{(2)}(t+\epsilon, t)$ be two Markovian Choi operators corresponding to two separate Markovian operation $\mathit{\Lambda}^{(1)}_{\mathcal{M}}$ and $\mathit{\Lambda}^{(2)}_{\mathcal{M}}$ having Lindblad type generator $\mathcal{L}^{(1)}_t$ and $\mathcal{L}^{(2)}_t$ with positive coefficients $\{\mathit{\Gamma}^{(1)}_k\}$ and $\{\mathit{\Gamma}^{(2)}_k\}$ respectively. Therefore we have, $\mathit{\Lambda}^{(i)}_{\mathcal{M}}(t+\epsilon, t) \equiv \mathbb{T}\exp\left(\int_t^{t+\epsilon} \mathcal{L}^{(i)}_t dt\right)$, with $i = 1, 2$. Let us now consider the maps $\mathit{\Lambda}^{(i)}_{\mathcal{M}}(t_2, t_1)$ for a finite time interval $t_2 - t_1 = N\epsilon$, where we devide the total time interval in large number of (N) snapshots of temporal width ϵ. Considering the limit $\epsilon \to 0$, we use the Lie Trotter formula [152] to get

$$
\begin{aligned}
&\mathit{\Lambda}^{(i)}_{\mathcal{M}}(t_2, t_1) = \\
&\mathbb{T}\exp\left(\sum_{\nu=1}^{N-1}\int_{t_\nu}^{t_{\nu+1}} \mathcal{L}^{(i)}_t dt\right) = \lim_{N\to\infty}\left[\mathit{\Pi}_{\nu=1}^{N-1}\mathbb{T}\exp\left(\frac{\int_{t_\nu}^{t_{\nu+1}}\mathcal{L}^{(i)}_t dt}{N}\right)\right]^N \\
&\approx \lim_{N\to\infty}\left[\mathbb{I} + \epsilon\frac{\sum_{t_\nu=t_1}^{t_2}\mathcal{L}^{(i)}_{t_\nu}}{N}\right]^N \approx \left[\mathbb{I} + \epsilon\sum_{t_\nu=t_1}^{t_2}\mathcal{L}^{(i)}_{t_\nu}\right].
\end{aligned}
$$

Here we use the following identity $\int_{t_\nu i}^{t_{\nu+1}} \mathcal{L}^{(i)}_{t_\nu} dt = \epsilon\mathcal{L}^{(i)}_{t_\nu}$, because $t_{\nu+1} - t_\nu = \epsilon \ \ (\forall \nu)$ and though the Lindblad operation is generally time dependent, in the very small snapshot of ϵ, the Lindbladian be considered as constant at that instant of time. We also restrict ourselves to first order of the Taylor series expansion of the exponential term and later the binomial expansion, in the limit $\epsilon \to 0$ and $\mathit{\Gamma}^{(i)}_k(t)\epsilon << 1 \ \ (\forall k)$.

$$\Lambda_{\mathcal{M}}(t_2,t_1) = p\Lambda^{(1)}_{\mathcal{M}}(t_2,t_1) + (1-p)\Lambda^{(2)}_{\mathcal{M}}(t_2,t_1)$$
$$= \mathbb{I} + \epsilon[p\sum_{t_\nu=t_1}^{t_2}\mathcal{L}^{(1)}_{t_{nu}} + (1-p)\sum_{t_\nu=t_1}^{t_2}\mathcal{L}^{(2)}_{t_\nu}] = \mathbb{I} + \epsilon\sum_{t_\nu=t_1}^{t_2}\mathcal{L}_{t_\nu},$$

with $\mathcal{L}_{t_\nu} = p\mathcal{L}^{(1)}_{t_\nu} + (1-p)\mathcal{L}^{(2)}_{t_\nu}$ and $0 \leq p \leq 1$, we get that $\mathcal{L}_{t_\nu}$ is also a Lindblad type generator with positive coefficients. This shows that the map $\Lambda_{\mathcal{M}}(t_2,t_1)$ also belongs to the set of divisible Markovian maps. If we now consider $t_2 = t+\epsilon$, the results holds true, because it is valid for all (t_1,t_2). This proves the statement $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is a convex set. □

Next we prove the following propositions where we establish that the free Choi operators in the context of resource theory of non-Markovianity also possess such properties.

Proposition 5A: Resourceful Choi operators cannot be generated under tensor product, partial trace and permutations of spatially separated subsystems.

Proof. Let us consider two arbitrary Choi operators $\rho^{\mathcal{M}}(t+\epsilon,t)$, $\sigma^{\mathcal{M}}(t+\epsilon,t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}$ (here we drop the index α of the Choi operators for brevity). Therefore $\parallel \rho^{\mathcal{M}}(t+\epsilon,t) \parallel_1 = \parallel \sigma^{\mathcal{M}}(t+\epsilon,t) \parallel_1 = 1$. Applying properties of trace norm [153] we obtain, $\parallel \rho^{\mathcal{M}}(t+\epsilon,t)\otimes\sigma^{\mathcal{M}}(t+\epsilon,t) \parallel_1 = \parallel \rho^{\mathcal{M}}(t+\epsilon,t) \parallel_1 \parallel \sigma^{\mathcal{M}}(t+\epsilon,t) \parallel_1 = 1$. We know that for a non-Markovian resourceful Choi operator, the trace norm must be strictly greater than 1 in some intermediate region. Therefore $\rho^{\mathcal{M}}(t+\epsilon,t)\otimes\sigma^{\mathcal{M}}(t+\epsilon,t)$ is not a resourceful Choi operator. Physical interpretation of this mathematical property is interesting. It states that even if there are two parallel processes running simultaneously and both of them are resourceless in nature, then the resulting process is again resourceless.

Similarly we can prove $\sigma^{\mathcal{M}}(t+\epsilon,t)\otimes\rho^{\mathcal{M}}(t+\epsilon,t)$ is also not resourceful.
To show partial trace cannot generate the resource, let $\rho^{\mathcal{M}}_{AB}(t+\epsilon,t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}$ be an arbitrary bipartite Choi operator. Taking partial trace with respect to the subsystem B, the reduced subsystem becomes $\rho^{\mathcal{M}}_{A}(t+\epsilon,t) = \mathrm{Tr}_B[\rho^{\mathcal{M}}_{AB}(t+\epsilon,t)]$. Now, $\parallel \rho^{\mathcal{M}}_{A}(t+\epsilon,t) \parallel_1 = \mathrm{Tr}[\sqrt{\rho^{\mathcal{M}}_{A}(t+\epsilon,t)\rho^{\dagger\mathcal{M}}_{A}(t+\epsilon,t)}] = \mathrm{Tr}[\rho^{\mathcal{M}}_{A}(t+\epsilon,t)]$, as $\rho^{\mathcal{M}}_{A}(t+$

$\epsilon, t)$ is a positive operator. Hence, we have

$$\rho_A^{\mathcal{M}}(t+\epsilon, t) = \sum_{j=1}^{d_B} (\mathbb{I}_A \otimes \langle b_i|)\rho_{AB}^{\mathcal{M}}(t)(\mathbb{I}_A \otimes |b_i\rangle), \tag{4.6}$$

where d_B is the dimension of subsystem B having orthonormal basis $\{|b_i|\}\rangle$. Therefore,

$$\mathrm{Tr}[\rho_A^{\mathcal{M}}(t+\epsilon,t)] = \mathrm{Tr}[\rho_{AB}^{\mathcal{M}}(t+\epsilon,t)(\mathbb{I}_A \otimes \textstyle\sum_{j=1}^{d_B} |b_i\rangle\langle b_i|)] = \mathrm{Tr}[\rho_{AB}^{\mathcal{M}}(t+\epsilon,t)(\mathbb{I}_A \otimes \mathbb{I}_B)]$$
$$= \mathrm{Tr}[\rho_{AB}^{\mathcal{M}}(t+\epsilon,t)] = 1.$$

So, we prove that $\rho_A^{\mathcal{M}}(t+\epsilon, t)$ is not a resourceful Choi operator. □

We re-emphasize that the picture which is constructed here is in terms of the Choi operators corresponding to the dynamical maps. Here in the previous proposition, when we prove that the tensor product of two free Choi operators is also free, we basically consider two different divisible processes acting separately on two separate states. Therefore, these two processes have absolutely no connection to each other, and hence, the total process is also divisible, if each of them are.

Proposition 5B: The set of free Choi operators $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ is a compact set.

Proof. A subset in an Euclidean space is compact if and only if it is bounded and closed (contains all its limit points). Boundedness of $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ in trace norm is clear from its definition and closedness of $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ follows from the continuity of trace norm. Hence $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ is compact. □

Proposition 5C: Free operations cannot generate a resourceful Choi operator

We have considered divisible operations as free operations. Applying a free operation on another free operation basically means concatenation of two divisible operations which gives rise to another single divisible operation. Therefore, the Choi operator corresponding to the resulting operation cannot be resourceful.

These propositions have the following implications. One can define free Choi operators in an arbitrary finite dimension. The tensor product of two free Choi operators and the reduced state of a free Choi operators cannot be resourceful [147], and free operations cannot generate the resource.

Measure of non-Markovianity: The minimum contractive distance between a non-Markovian operation and the set of all Markovian operations gives rise to a proper measure of non-Markovianity. The CJ isomorphism [6,7] reduces this problem to finding the distance between the corresponding Choi operators. A measure of non-Markovianity can be defined as $M_T(\mathcal{C}^{\mathcal{N}}(t+\epsilon,t)) = \inf_{\mathcal{C}_\alpha^{\mathcal{M}}(t+\epsilon,t)} D(\mathcal{C}^{\mathcal{N}}(t+\epsilon,t)|\mathcal{C}_\alpha^{\mathcal{M}}(t+\epsilon,t))$, where $D(\cdot|\cdot)$ is any metric contractive under CPTP maps. $M_T(\mathcal{C}^{\mathcal{N}}(t+\epsilon,t))$ can only be a non-zero positive quantity in the time span ϵ, where CP breaks down. The optimization is carried out over the free Choi operators $\mathcal{C}_\alpha^{\mathcal{M}}(t+\epsilon,t)$. In general this is extremely difficult to calculate because the free operations $\Lambda^{\mathcal{M}}$ do not form a convex set [150,151]. By virtue of the following arguments and proposition we overcome this difficulty.

It can be shown that in the limit of $\epsilon \to 0$, the set of all Choi operators $\mathcal{A}$ is also a convex set. Using **Proposition 4** (4.1), we define the following measure of non-Markovianity, taking the right derivative of $M_T(t+\epsilon,t)$ in the CP breaking region as

$$\mathcal{D}_T(t) = \lim_{\epsilon\to 0^+} \frac{M_T(t+\epsilon,t) - 0}{\epsilon}$$

, where $M_T(t+\epsilon,t) = \inf_{\mathcal{C}_\alpha^{\mathcal{M}}(t+\epsilon,t)} ||(\mathcal{C}^{\mathcal{N}}(t+\epsilon,t) - \mathcal{C}_\alpha^{\mathcal{M}}(t+\epsilon,t)||_1$, with $||\cdot||_1$ is the trace norm. As divisibility is not broken before t,the minimum distance M at the previous time t is taken to be zero and the Choi operator at that time belongs to the set of free Choi operators. For any quantum evolution, we always have $\mathcal{D}_T(t) \geq 0$. The equality holds for divisible Markovian evolutions. The optimization involved in evaluating $\mathcal{D}_T(t)$ becomes relatively easier because the set of free Choi operators now forms a convex set. Moreover, by virtue of **Proposition 5B** (4.1) and **Proposition 4** (4.1), we know that the set of free Choi operators $\mathbb{F}$ in the limit $\epsilon \to 0$ is convex and compact. We can thus apply the Krein-Milman theorem [154] to state that $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ is the convex hull of its extreme points. Hereafter, for brevity we use the short hand notation $\mathcal{C}^{\mathcal{N}}(t+\epsilon,t) = \mathcal{C}^{\mathcal{N}}(t)$ and $\mathcal{C}_\alpha^{\mathcal{M}}(t+\epsilon,t) = \mathcal{C}^{\mathcal{M}}(t)$

In quantum theory, a bona fide resource measure [127] should satisfy faithfulness, convexity and a monotonicity under the free operations corresponding to that resource theory. In case non-Markovianity, faithfulness of a measure means that the resource quantifier will give non-zero value iff an operation is resourceful. Since we are discussing a convex resource theory, convexity of the resource measure follows naturally. Finally, monotonicity under free operations is the next important property of a good resource quantifier. For the resource theory of non-Markovianity, monotonicity stems from the fact that concatenating one or more free operations to the original operation cannot increase the resource, and hence, the resource quantifier should be non increasing under free operations.

Proposition 6: *$\mathcal{D}_T(t)$ is a bona fide measure of non-Markovianity.*

Proof. To prove $\mathcal{D}_T(t)$ is a bona fide measure, we have to prove it is faithful, convex and a monotone under the free operations.

The measure is faithful iff $\mathcal{D}_T(t) \geq 0$ and $\mathcal{D}_T(t) = 0 \Leftrightarrow \mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}$. Clearly $\mathcal{D}_T(\mathcal{C}^{\mathcal{Q}}(t)) \geq 0 \; \forall \mathcal{C}^{\mathcal{Q}}(t)$ from the definition. We now show that $\mathcal{D}_T(\mathcal{C}^{\mathcal{M}}(t)) = 0$ iff $\mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}$. Here the if part is obvious from the definition. To prove the only if part, let $\mathcal{D}_T(\mathcal{C}^{\mathcal{Q}}(t)) = 0$ for some $\mathcal{C}^{\mathcal{Q}}(t)$. Then $\mathcal{D}_T(\mathcal{C}^{\mathcal{Q}}(t)) = 0 \Rightarrow \lim_{\epsilon \to 0} \dfrac{M_T(\mathcal{C}^{\mathcal{Q}}(t))}{\epsilon}$ $= 0 \Rightarrow M_T(\mathcal{C}^{\mathcal{Q}}(t)) = 0$, since ϵ is a finite positive number. Therefore $M_T(\mathcal{C}^{\mathcal{Q}}(t)) \Rightarrow$ $\inf_{\mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}} D(\mathcal{C}^{\mathcal{Q}}(t) | \mathcal{C}^{\mathcal{M}}(t)) = 0$. This implies $\mathcal{C}^{\mathcal{Q}}(t) \in \mathrm{Cl}(\mathbb{F}) = \mathbb{F}$ since $\mathbb{F}$ is a closed set. Here Cl(.) stands for the topological closure. Therefore $\mathcal{D}_T(t)$ is faithful.

To prove the convexity of $\mathcal{D}_T(t)$, we consider $\mathcal{C}_1^{\mathcal{N}}(t)$ and $\mathcal{C}_2^{\mathcal{N}}(t)$ be two Choi operators. By the convexity property of the set of all Choi operators $\mathcal{A}$, we know $\mathcal{C}^{\mathcal{N}}(t) = p\mathcal{C}_1^{\mathcal{N}}(t) + (1-p)\mathcal{C}_2^{\mathcal{N}}(t)$ is also a Choi operator. Therefore, from the triangle inequality, we have $||p\mathcal{C}_1^{\mathcal{N}}(t) + (1-p)\mathcal{C}_2^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t)||_1 \leq p||\mathcal{C}_1^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t)||_1 + (1-p)||\mathcal{C}_2^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t)||_1$, for all $\mathcal{C}^{\mathcal{M}}(t)$. Consequently, we get $\inf_{\mathcal{C}^{\mathcal{M}}(t)} ||p\mathcal{C}_1^{\mathcal{N}}(t) + (1-p)\mathcal{C}_2^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t)||_1 \leq p \inf_{\mathcal{C}^{\mathcal{M}}(t)} ||\mathcal{C}_1^{\mathcal{N}}(t) - \mathcal{C}_1^{\mathcal{M}}(t)|| + (1-p) \inf_{\mathcal{C}^{\mathcal{M}}(t)} ||\mathcal{C}_2^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t)||_1$. This in turn proves the convexity relation $\mathcal{D}_T(p\mathcal{C}_1^{\mathcal{M}}(t) + (1-p)\mathcal{C}_2^{\mathcal{M}}(t)) \leq p\mathcal{D}_T(\mathcal{C}_1^{\mathcal{M}}(t)) + (1-p)\mathcal{D}_T(\mathcal{C}_2^{\mathcal{M}}(t))$.

To prove the monotonicity of $\mathcal{D}_T(t)$, we consider a divisible free operation: $\rho(t_2) = \mathbb{I} \otimes \Lambda(t_2, t_1)(\rho(t_1)) = \mathrm{Tr}_E \left[V(t_2, t_1)\rho(t_1) \otimes \sigma_E V^{\dagger}(t_2, t_1) \right]$, where $V(t)$ is a global unitary acting on the composite system-environment Hilbert space and

σ_E is the initial state of the environment. Therefore, we have $||\mathcal{C}^{\mathcal{N}}(t+\Delta) - \mathcal{C}^{\mathcal{M}}(t+\Delta)||_1 = ||\mathrm{Tr}_E\left[V(t+\Delta,t)(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))\otimes\sigma_E V^\dagger(t+\Delta,t)\right]||_1$. Using the trace norm inequality $||\mathrm{Tr}_b[A_{ab}]||_1 \leq ||[A_{ab}]||_1$, for any bounded operator A_{ab}, and preservation of trace norm under unitary rotation, we thus have $||(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))||_1 = ||(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))\otimes\sigma_E||_1 = ||V(t+\Delta,t)(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))\otimes\sigma_E V^\dagger(t+\Delta,t)||_1 \leq ||\mathrm{Tr}_E[V(t+\Delta,t)(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))\otimes\sigma_E V^\dagger(t+\Delta,t)]||_1 = ||(\mathcal{C}^{\mathcal{N}}(t+\Delta) - \mathcal{C}^{\mathcal{M}}(t+\Delta))||_1$. Therefore we have $||(\mathcal{C}^{\mathcal{N}}(t+\Delta) - \mathcal{C}^{\mathcal{M}}(t+\Delta))||_1 \leq ||(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))||_1$.

Now $M_T(t',t) = \inf_{\mathcal{C}^{\mathcal{M}}(t)} ||\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t)||_1 = ||\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}*}(t)||_1$, with $\mathcal{C}^{\mathcal{M}*}(t)$ being the free Choi operator from which the distance is minimum. Using the fact $\mathbb{I}\otimes\Lambda(t+\Delta,t)(\mathcal{C}^{\mathcal{M}*}(t) \in \mathbb{F}$, we have $\inf_{\mathbb{I}\otimes\Lambda(t+\Delta,t)(\mathcal{C}^{\mathcal{M}*}(t)} ||\mathbb{I}\otimes\Lambda(t+\Delta,t)(\mathcal{C}^{\mathcal{N}}(t)) - \mathbb{I}\otimes\Lambda(t+\Delta,t)(\mathcal{C}^{\mathcal{M}*}(t))||_1 \leq ||\mathbb{I}\otimes\Lambda(t+\Delta,t)(\mathcal{C}^{\mathcal{N}}(t)) - \mathbb{I}\otimes\Lambda(t+\Delta,t)(\mathcal{C}^{\mathcal{M}*}(t))||_1 \leq ||\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}*}(t)||_1$. It is evident from Eq. (4.1) that $\mathcal{D}_T(t+\Delta) \leq \mathcal{D}_T(t)$, proving the monotonicity of $\mathcal{D}_T(t)$ under divisible operations. This completes the proof of the proposition. □

It is to be noted that the complexity of calculating the measure of non-Markovianity can be further reduced by constructing a lower bound of $\mathcal{D}_T(t)$ in the following theorem.

Theorem 7: Let $\Lambda^{\mathcal{N}}$ be a map corresponding to some operation $\mathcal{N}$ and $g^{\mathcal{N}}(t)$ be the RHP measure, then $\mathcal{D}_T(t)$ is bounded below by $g^{\mathcal{N}}(t)$, *i.e.* $\mathcal{D}_T(t) \geq g^{\mathcal{N}}(t)$.

Proof. We have the expression of our non-Markovianity measure

$$\mathcal{D}_T(t) = \lim_{\epsilon\to 0^+} \frac{\inf_{\mathcal{C}^{\mathcal{M}}} \parallel \mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t) \parallel_1}{\epsilon}.$$

Using the reverse triangle inequality: $||A-B||_1 \geq |\parallel A \parallel_1 - \parallel B \parallel_1|$ with $\parallel \mathcal{C}^{\mathcal{N}}(t) \parallel_1 \geq 1$ and $\parallel \mathcal{C}^{\mathcal{M}}(t) \parallel_1 = 1 \;\; \forall \Lambda_{\mathcal{M}}$, we find

$$\mathcal{D}_T(t) \geq \lim_{\epsilon\to 0^+} \frac{\parallel \mathcal{C}^{\mathcal{N}}(t) \parallel_1 - 1}{\epsilon} = g^{\mathcal{N}}(t)$$

□

Interestingly, $\mathcal{D}_T(t)$ is lower bounded by $g^{\mathcal{N}}(t)$, which is free from optimization and easier to calculate. Note that $g^{\mathcal{N}}(t)$ is the time derivative of the trace norm of the Choi operator [80]. When the divisibility breaks down, the norm of the corresponding Choi operator is strictly greater that 1. In those regions we have $g^{\mathcal{N}}(t) > 0$. This establishes RHP measure as a witness of CP-indivisibility, whereas $\mathcal{D}_T(t)$ is the time derivative of the minimum distance between the Choi operator corresponding to a specific evolution and all possible free Choi operators.

Corollary: The RHP measure $g^{\mathcal{N}}(t)$ is also a bona fide measure of non-Markovianity.

Proof. The faithfullness of $g^{\mathcal{N}}(t)$ follows from its original definition [80]. The convexity of $g^{\mathcal{N}}(t)$ follows from the convexity of $||\mathcal{C}_{\mathcal{N}}(t)||_1$, as proved in **Proposition 6** (4.1). The monotonicity of $g^{\mathcal{N}}(t)$ follows from the trace inequality $||Tr_b[A_{ab}]||_1 \leq ||[A_{ab}]||_1$, for any bounded operator A_{ab}, as done in **Proposition 6** (4.1). □

4.2 Robustness of non-Markovianity

We now propose the idea of robustness of non-Markovianity (RONM). The concept is similar to the robustness of the resource theories of entanglement [155–157], coherence [158] and asymmetry [159]. In accordance with the definitions of robustness for other quantum resources, we define RONM as the minimum amount of noise (Markovian or non-Markovian) needed to be added to a non-Markovian evolution to make the resulting evolution Markovian. Hence, the formal definition of RONM is given by

$$\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t)) = \inf_{s} \left\{ s \geq 0 : \frac{\mathcal{C}^{\mathcal{N}}(t) + s\tau^{\mathcal{N}}(t)}{1+s} = \delta^{\mathcal{M}}(t) \in \mathbb{F} \right\}, \tag{4.7}$$

where $\tau^{\mathcal{N}}(t)$ is an arbitrary element from the set of all Choi operators $\mathcal{A}$. Achieving the minimization for Choi operators $\tau^{\mathcal{N}*(t)}$ and $\delta^{\mathcal{M}*(t)}$, we write

$$\mathcal{C}^{\mathcal{N}}(t) = [1 + \mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))]\delta^{\mathcal{M}*}(t) - \mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))\tau^{\mathcal{N}*}(t). \tag{4.8}$$

We would like to mention here that in a recent work [160] another definition of robustness of non-Markovianity has also been proposed. However, the authors have considered only the divisible operations over which the optimization is based on. It is important to mention here the fact that convex mixing of a non-markovian noise with another non-Markovian dynamics can make it Markovian [161]. Recently the same has been demonstrated experimentally [162]. Therefore the definition provided in [160] may of course be suitable for certain situations but it lacks generality, since it excludes a considerable portion of noise in the definition.

We now prove the following proposition where we establish that $\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))$ is also a very useful measure of non-Markovianity.

Proposition 7: $\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))$ is a faithful, convex measure of non-Markovianity and lower bounded by $\frac{1}{2}\int_0^t g^{\mathcal{N}}(t')dt'$.

Proof. The faithfulness of RONM follows from the definition as

$$\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{Q}}(t)) \geq 0 \;\; \text{and} \;\; \mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{Q}}(t)) = 0 \Leftrightarrow \mathcal{C}^{\mathcal{Q}}(t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}.$$

To prove the convexity of $\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))$, let us consider two arbitrary Choi operators $\mathcal{C}_1^{\mathcal{N}}(t)$ and $\mathcal{C}_2^{\mathcal{N}}(t)$, expressed as the pseudo-mixture $\mathcal{C}_l^{\mathcal{N}}(t) = [1+\mathcal{R}_{\mathcal{N}}(\mathcal{C}_l^{\mathcal{N}}(t))]\delta_l^{\mathcal{M}*}(t) - \mathcal{R}_{\mathcal{N}}(\mathcal{C}_l^{\mathcal{N}}(t))\tau_l^{\mathcal{N}*}(t)$ (for $l = 1, 2$). The convex structure of $\mathcal{A}$ ensures that the convex combination of these two Choi operators will also be another Choi operator. Now considering the convex decomposition $\mathcal{C}^{\mathcal{N}}(t) = p\mathcal{C}_1^{\mathcal{N}}(t) + (1-p)\mathcal{C}_2^{\mathcal{N}}(t)$ (with $0 \leq p \leq 1$) and utilizing the pseudo-mixtures written above, the following pseudo-mixture $\mathcal{C}^{\mathcal{N}}(t) = [1+s]\delta^{\mathcal{M}}(t) - s(\mathcal{C}^{\mathcal{N}}(t))\tau^{\mathcal{N}}(t)$ can be written with $\delta^{\mathcal{M}}(t) = \{p[1+\mathcal{R}_{\mathcal{N}}(\mathcal{C}_l^{\mathcal{N}}(t))]\delta_l^{\mathcal{M}*}(t) + (1-p)[1+\mathcal{R}_{\mathcal{N}}(\mathcal{C}_2^{\mathcal{N}}(t))]\delta_2^{\mathcal{M}*}(t)\}/(1+s)$ and $\tau^{\mathcal{N}}(t) = \{p\mathcal{R}_{\mathcal{N}}(\mathcal{C}_l^{\mathcal{N}}(t))\tau_l^{\mathcal{N}*}(t) + (1-p)\mathcal{R}_{\mathcal{N}}(\mathcal{C}_2^{\mathcal{N}}(t))\tau_2^{\mathcal{N}*}(t)\}/s$. The trace preservation property of any arbitrary quantum operation guarantees the Hermiticity of Choi operator with unit trace. Therefore, from the normalization condition, we get $s = p\mathcal{R}_{\mathcal{N}}(\mathcal{C}_1^{\mathcal{N}}(t)) + (1-p)\mathcal{R}_{\mathcal{N}}(\mathcal{C}_2^{\mathcal{N}}(t))$. Now, from the definition of RONM, we have $\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t)) \leq s$. Thus, the convexity of $\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))$ is proved.

To find the lower bound of $\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))$, we take the help of a result from entanglement theory [163]. As we have seen from Eq. (4.8), the Choi operator $\mathcal{C}^{\mathcal{N}}(t)$ can be written in the pseudo mixture $\mathcal{C}^{\mathcal{N}}(t) = c_+\delta^{\mathcal{M}*} - c_-\tau^{\mathcal{N}*}$, with $c_+ = 1+$

$\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))$ and $c_- = \mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))$. Now from [163] we know, that for the Hermitian matrix $\mathcal{C}^{\mathcal{N}}(t)$, there exist a minimal decomposition $\mathcal{C}^{\mathcal{N}}(t) = a_+\rho^+ - a_-\rho^-$, such that $||\mathcal{C}^{\mathcal{N}}(t)||_1 = a_+ + a_-$ is minimum and ρ^+, ρ^- has disjoint support.

Therefore, if $\mathcal{P}_-$ be the projector onto the negative eigenvalue subspace of $\mathcal{C}^{\mathcal{N}}(t)$, then $a_- = -\mathrm{Tr}[\mathcal{P}_-\mathcal{C}^{\mathcal{N}}(t)]$ is the sum of absolute values of negative eigenvalues [163]. Let us consider the spectral decomposition $\mathcal{C}^{\mathcal{N}}(t) = \sum_k \lambda_k^+|\lambda_k^+\rangle + \sum_l \lambda_l^-|\lambda_l^-\rangle$ with λ_k^+s and λ_l^-s being the positive and negative eigenvalues respectively. Therefore, the trace preservation condition yields $\sum_k \lambda_k^+ = 1 + \sum_l |\lambda_l^-|$. Thus, we have $||\mathcal{C}^{\mathcal{N}}(t)||_1 = \sum_k \lambda_k^+ + \sum_l |\lambda_l^-| = 1 + 2\sum_l |\lambda_l^-|$. Since from [163] we know $\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t)) = c_- \geq a_- = \sum_l |\lambda_l^-|$, we get the following inequality

$$\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t)) \geq \frac{||\mathcal{C}^{\mathcal{N}}(t)||_1 - 1}{2}. \tag{4.9}$$

Now because of the fact that $g^{\mathcal{N}}$ is the right derivative of the trace norm of Choi operator, we have

$$\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t)) \geq \frac{1}{2}\int_0^t g^{\mathcal{N}}(t')dt' = \frac{1}{2}\mathcal{N}_T(t) \quad \text{(say)} \tag{4.10}$$

This completes the proposition. □

We note that the RONM is lower bounded by the integral of RHP measure $g^{\mathcal{N}}(t)$. This provides a physical interpretation of the RHP measure. RONM physically means the endurance of a non-Markovian operation under mixing with arbitrary noise. The normalized measure of non-Markovianity [80] can therefore be found to satisfy the following inequality

$$\mathcal{T}(t) = \frac{\mathcal{N}_T(t)}{1+\mathcal{N}_T(t)} \leq \frac{2\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))}{1+2\mathcal{R}_{\mathcal{N}}(\mathcal{C}^{\mathcal{N}}(t))}.$$

Finally, as an application of resourse theory of non-Markovianity, we consider the non-Markovianity induced improvement of quantum capacity for dephasing channels. In earlier works [99, 164] it has been established that the quantum capacity of a dephasing channel can be represented by $\mathcal{Q}_D(t) = 1 - H_2(\frac{1+e^{-G(t)}}{2})$, where $H_2(\cdot)$ is the binary Shannon entropy and $G(t) = \frac{1}{2}\int_0^t \Gamma(t')dt'$, for a qubit dephasing channel $\dot{\rho}(t) = \frac{\Gamma(t)}{2}(\sigma_z \rho(t) \sigma_z - \rho(t))$. For a non-Markovian dephasing channel, the quantum capacity satisfies the following inequality

$$\mathcal{Q}_D(t) \leq 1 - H_2\left(\frac{1 + e^{2\mathcal{R}_{\mathcal{N}}^D} e^{-G_M(t)}}{2}\right), \tag{4.11}$$

where $G_M(t) = \frac{1}{2}\int_{\Gamma(t')\geq 0} \Gamma(t')dt'$, and $\mathcal{R}_{\mathcal{N}}^D$ is the robustness of the non-Markovian dephasing channel. Since $G_M(t) - \mathcal{N}_T^D(t) = G(t) \geq 0$ with $\mathcal{N}_T^D(t) = \frac{1}{2}\int_{\Gamma(t')<0} \Gamma(t')dt'$, the total exponential term on the right hand side $e^{-G(t)} \leq 1$. However, if the evolution is sufficiently indivisible, so that $G(t) \rightarrow 0$, even for a sufficiently long time evolution, the capacity of the channel improves tremendously. This is indeed possible in practical situations, where the dephasing coefficient takes the form $\Gamma(t) \sim \tan(t)$ [80]. For these evolutions, we have $G(t) = 1 - \ln(\cos(t))$, which can vanish even for sufficiently large instances of time, and hence, $\mathcal{Q}_D(t) = 1$ for such situations. Therefore, near perfect communication can be achieved through lossy quantum channels, using non-Markovianity as a resource.

As a concluding note of this chapter, we have constructed a resource theory of non-Markovianity. The resource theory is a convex under small time interval approximation. Being a dynamics resource, we have identified resourceless operations in terms of information backflow from the environment to the system. This in turn characterizes the resourceful operations in this context. A measure of non-Markovianity has been constructed. This measure may be hard to compute but it is lower bounded by well known RHP measure of non-Markovianity. Therefore any non-Markovian operation detected by RHP measure can also be detected via our proposed measure. We finally construct robustness of non-Markovianity. We have shown robustness is a faithful measure of non-Markovianity. This is operationally motivated. We have also link robustness with RHP measure. This construction automatically raises the question of witnessing non-Markovianity. In the next chapter we address the idea of non-Markovianity witness.

Chapter 5

Convex geometry of Markovian Lindblad dynamics and witnessing non-Markovianity

One of the important questions from the perspective of any resource theory is how to certify the resource. Ideally the certification of resource is done by separating the resource from resourceless objects. Hence there should be something which will create a separation between these two. This is the concept of resource witness. In case of entanglement theory the idea of entanglement witness was first proposed in [32]. As separable or resourceless states form a convex and compact subset of the state space, tools from convex geometry can be invoked to construct the entanglement witness. Precisely hyperplane separation provides a beautiful way to idealise the witness.We would like to ask the same question in the context of non-Markovianity. Non-Markovianity is a dynamic resource whose resource theory has been formulated in the previous chapter. Now we would like to separate out a resourceful operation from resourceless Markovian *i.e.* CP-divisible operations. The main obstacle to formulate the idea of non-Markovianity witness in the similar way like entanglement witness is that the set of Markovian operations do not form a convex set in general [150, 151]. In the previous chapter we have shown that this difficulty can be overcome under the idea of small time interval approximation. Therefore we obtain a convex and compact structure of resourceless operations under small time interval approximation. This provides us a platform to exploit convex geometry tools to study the idea of non-Markovianity witness. Now we shall discuss in details on detecting non-Markovianity by the notion of non-Markovianity witness. This chapter is based on the work [148].

5.1 Theory of linear witnesses for non-Markovianity detection

In this section we develop the main theory of linear witnesses to detect non-Markovian dynamics, based on the structure of the Choi operator corresponding to dynamical maps having Lindblad type generators.

Proposition 8: The set of Markovian Choi operators $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ (4.5) is a convex and compact set in the limit $\epsilon \to 0$ and $\Gamma_k(t)\epsilon << 1 \quad (\forall k)$.

The proof of the proposition has already been discussed in the previous chapter (4.1). In light of this proposition we would like to make following comment.

As we have shown that the proposition holds good even if we consider many snapshots of the width ϵ. We have considered here a single snapshot at a particular time. But if need be, this proposition can be extended to a much larger time gap with many infinitesimally small snapshots.

From **Proposition 8**, it can be concluded that convex structure comes only under the small time interval approximation, by utilizing the structure of the Lindbladians. If experimental perspective is concerned, it requires observation in snapshots taken at very frequent intervals in the dynamical evolution.

We have already mentioned that Quantum non-Markovianity is a dynamical property. Whether the dynamics exhibits information backflow, can only be found if the specific information (*e.g.* entanglement, distinguishability) of the time evolved state is compared with that of a state at a previous or later time and hence a fixed time period thus comes naturally and unavoidably into the picture. Here we are fixing the range of this snap shots in terms of a fixed ϵ. However, the exact value of ϵ is of course determined by specific experimental situations.

Since we are dealing with finite dimensional normed linear spaces only, all norms are topologically equivalent. In light of this fundamental fact, evidently $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is also compact under Hilbert-Schmidt (HS) norm $||\cdot||_2 = \sqrt{\mathrm{Tr}[(\cdot)^{\dagger}(\cdot)]}$. Thus the structure of $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ leads us to the proof of the existence of linear non-Markovianity witnesses.We take the help of geometric Hahn-Banach separation theorem [61].

Theorem 8: A non-Markovian Choi operators can be separated from all Markovian Choi operators by a hyperplane.

Proof: Let $\mathcal{C}^{\mathcal{N}}(t)$ be a non-Markovian Choi operator and $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ denotes the set of all Markovian Choi operators. Note that, $D(\mathcal{C}^{\mathcal{N}}(t) \mid \mathbb{F}^{\epsilon}_{\mathcal{M}}) = 0$ iff $\mathcal{C}^{\mathcal{N}}(t) \in \mathrm{Cl}(\mathbb{F}^{\epsilon}_{\mathcal{M}})$, where $\mathrm{Cl}(\cdot)$ denotes the topological closure and $D(.)$ is any metric. Since $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is closed as $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is compact, then $\mathrm{Cl}(\mathbb{F}^{\epsilon}_{\mathcal{M}}) = \mathbb{F}^{\epsilon}_{\mathcal{M}}$. $\mathcal{C}^{\mathcal{N}}(t)$ does not belong to $\mathbb{F}^{\epsilon}_{\mathcal{M}}$, thus $D(\mathcal{C}^{\mathcal{N}}(t) \mid \mathbb{F}^{\epsilon}_{\mathcal{M}}) > 0$. Considering singleton set $\{\mathcal{C}^{\mathcal{N}}(t)\}$ as convex set, we always have a hyperplane [61] separating $\mathcal{C}^{\mathcal{N}}(t)$ and $\mathbb{F}^{\epsilon}_{\mathcal{M}}$. $\square$

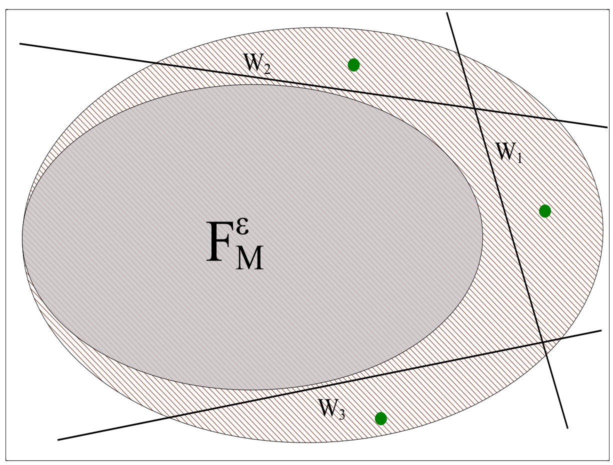

Figure 5.1: (Colour online) Here the larger ellipse represents set of all Choi states and $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ represents the convex compact set of MCS at sufficiently small time ϵ and W_1, W_2, W_3 represents three hyperplanes separating three non-Markovian Choi operators (represented as green dot) from Markovian Choi operators.

In Fig. 5.1 we represent the schematic diagram for **Theorem 8**. Since $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is convex and compact, every non-Markovian Choi operators can in principle be separated from $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ by some separating hyperplane.

5.1.1 Non-Markovianity Witness:

Let us consider the construction of non-Markovianity witness using the techniques of entanglement theory.

Definition: A hermitian operator W is said to be a non-Markovianity witness if it satisfies following criteria:

1. $\mathrm{Tr}(W\mathcal{C}^{\mathcal{M}}(t)) \geqslant 0 \; \forall \; \mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}$.

2. There exists atleast one non-Markovian Choi state $\mathcal{C}^{\mathcal{N}}(t)$ such that $\mathrm{Tr}(W\mathcal{C}^{\mathcal{N}}(t)) < 0$.

We have to mention that a single witness can not detect all non-Markovian Choi operators. The witness will depend on the non-Markovian Choi operator, which one wishes to detect. This is also similar to that of the theory of entanglement witness.

Before constructing non-Markovianity witness let us illustrate the idea in brief. The total evolution is of course completely positive and thus the Choi state corresponding to a complete evolution starting from $t = 0$ to any later time t will always be a valid quantum state. Divisibility breaks down when complete positivity breaks within the dynamics; *e.g.* from some intermidiate time t to $t + \epsilon$. Our aim is to capture this breaking of divisibility via constructing a witness by a given Hermitian operator. For the very purpose we take a maximally entangled state and apply a given quantum channel on one party of this bipartite system. The witnessing. *i.e.* measuring the W on this bipartite system is done repeatedly at a very small time interval ϵ and the corresponding measurement data is taken over these various time intervals. In a sense this is similar to the determination of non-Marvovianity by information backflow. Now let us discuss the construction of non-Markovianity witness.

Construction of non-Markovianity witness

Let $\mathcal{C}^{\mathcal{Q}}(t)$ be a finite dimensional Choi operator corresponding to some operation $\mathcal{Q}$. As $\mathcal{C}^{\mathcal{Q}}(t)$ is Hermitian, we have its spectral decomposition as:

$$\mathcal{C}^{\mathcal{Q}}(t) = \sum \lambda_i P_i, \tag{5.1}$$

where P_is are orthogonal projections onto the subspace spanned by the normalised eigenvectors corresponding to the eigenvalues λ_i. Note that, $Tr(\mathcal{C}^{\mathcal{Q}}(t)P_j) = \lambda_j \delta_{ij}$, with δ_{ij} being the Kronecker delta. If the operation is CP-divisible, then $\mathcal{C}^{\mathcal{Q}}(t)$ being a valid quantum state has all non negative eigenvalues. Hence, $Tr(\mathcal{C}^{\mathcal{Q}}(t)P_j)$ $\geqslant 0 \ \forall$ j. If the operation is non-Markovian, $Tr(\mathcal{C}^{\mathcal{Q}}(t)P_j) < 0$ for atleast one j, as $\mathcal{C}^{\mathcal{Q}}(t)$ has atleast one negative eigenvalue. Thus the orthogonal projectors serve as witnesses for non-Markovianity.

Let us present two examples of non-Markovianity witness based on this construction.

Examples of non-Markovianity witness

Qubit dephasing channel: The Lindblad equation corresponding to such a channel is: $\dfrac{d\rho}{dt} = \gamma(t)(\sigma_i \rho \sigma_i - \rho)$, where σ_i $(i = x, y, z)$ are the Pauli matrices.

Performing small time approximation, we get the corresponding Choi matrix:

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} - \gamma(t)\epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} - \gamma(t)\epsilon & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

eigenvalues of the corresponding Choi operators are 0, 0, $\gamma(t)\epsilon$ and $1 - \gamma(t)\epsilon$ respectively. The dynamics may not be completely known always, but in some situations where one can estimate the numerical value of the decoherence rate *i.e.* $\gamma(t)$ or its upper bound can be estimated [165]. Since ϵ can be chosen infinitesimally small, depending on the estimation of the decoherence rate one can have $|\gamma(t)\epsilon| << 1$. Therefore for Markovian dynamics the later of the two non zero eigenvalues can always be considered as positive. We therefore take the orthogonal projector corresponding to the eigenvalue $\gamma(t)\epsilon$ as the witness for non-Markovianity since $\gamma(t) \geq 0$ for CP-divisible operations and can be negative in case of non-Markovian operations. The witness for qubit dephasing operation is therefore $P_{deph} = |\chi\rangle \langle\chi|$, where $|\chi\rangle = (-1,\ 0,\ 0,\ 1)^T$. This suggests that one has to perform single projective measurement onto $|\chi\rangle$ state to detect non-Markovianity of qubit dephasing channel.

Qubit Pauli Channel: Consider now the Pauli channel: $\frac{d\rho}{dt} = \gamma_x(t)(\sigma_x\rho\sigma_x - \rho)+\gamma_y(t)(\sigma_y\rho\sigma_y - \rho)+\gamma_z(t)(\sigma_z\rho\sigma_z - \rho)$. Small time approximation gives the eigenvalues of the corresponding Choi operators as $1 - (\gamma_x(t) + \gamma_y(t) + \gamma_z(t))\epsilon$, $\gamma_x(t)\epsilon$, $\gamma_y(t)\epsilon$, $\gamma_z(t)\epsilon$. Following the similar logic,we consider the orthogonal projectors corresponding to the eigenvalues $\gamma_i(t)\epsilon$ (with $i = x, y, z$) as the witnesses of non-Markovianity. They are respectively given by $P_x = |\chi\rangle_x \langle\chi|_x$, $P_y = |\chi\rangle_y \langle\chi|_y$ and $P_z = |\chi\rangle_z \langle\chi|_z$, with $|\chi\rangle_x = (0,\ 1,\ 1,\ 0)^T$, $|\chi\rangle_y = (0, -1,\ 1,\ 0)^T$ and $|\chi\rangle_z = (-1,\ 0,\ 0,\ 1)^T$.

It is worthwhile to mention that for the class of qubit dephasing and Pauli channels, the eigenvectors are independent of the channel parameters, as we can see from the example. The witness hence, can wit non-Markovianity for any qubit dephasing channels and for the second example, any qubit Pauli channels. Therefore, for the examples we have shown, the experimenter must have the prior knowledge about the type of the channel under study i.e whether the channel is dephasing or Pauli channel, to wit its non-Markovianity.

We would like to make a very important remark here. Non-Markovianity obtained from the second example that we have considered can be used to express something known as "eternal" non-Markovian dynamics in the literature [166]. For example let us consider a dynamics given by

$$\frac{d\rho}{dt} = \gamma_x(t)(\sigma_x\rho\sigma_x - \rho) + \gamma_y(t)(\sigma_y\rho\sigma_y - \rho) + \gamma_z(t)(\sigma_z\rho\sigma_z - \rho),$$

where $\gamma_x(t) = \gamma_y(t) = 1$ and $\gamma_z(t) = -tanh\ t$

The dynamics is non-Markovian in nature, since one of the Lindblad coefficient is always negative. But inspite of being non-Markovian in nature, it can not be witnessed by various distance, volume or entanglement based measures [166, 167] . But using our proposed method, we can construct the witnesses of Pauli channel (as done in the previous example) and detect the non-Markovian nature of the dynamics. Thus, for identifying such kind of non-Markovian dynamics, our method of linear witnesses has a clear advantage.

5.2 Detecting non-Markovianity of various qubit channels by Bell measurement:

In this section we shall provide examples of qubit non-Markovianity witness based on Bell measurement. These examples are operationally motivating, as without knowing the the dynamics completely one can detect whether it is Markovian or not by measuring in the Bell basis. Two qubit Bell states are given by

$$|\psi\rangle_{1,2} = \frac{1}{\sqrt{2}}\left[|00\rangle \pm |11\rangle\right]$$

and

$$|\psi\rangle_{3,4} = \frac{1}{\sqrt{2}}\left[|01\rangle \pm |10\rangle\right]$$

where $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

It is to be noted that for a Markovian operation, the corresponding Choi operator is a valid density matrix i.e it is Hermitian, positive and its trace is 1. Hence whenever we take expectation of a Markovian Choi operator with respect to any Bell state it give a non-negative value. This helps us to detect non-Markovianity of several qubit dynamics. Let us illustrate this fact via some examples.

Pauli channels: The Lindblad equation for any individual Pauli channel is given by

$$\frac{d\rho}{dt} = \gamma_i(t)(\sigma_i\rho\sigma_i - \rho) = \mathcal{L}_i^p(\rho)$$

where $i = x, y, z$ Here after considering small time interval approximation, the action of the corresponding channel can be expressed as: $\Lambda_i^p(\rho) \approx (\mathbb{I} + \epsilon\mathcal{L}_i^p)(\rho)$.

Qubit dephasing channel in σ_z basis: The Lindblad equation of dephasing channel in σ_z basis is given by

$$\frac{d\rho}{dt} = \mathcal{L}_z^p(\rho) = \gamma_z(t)(\sigma_z\rho\sigma_z - \rho) = \mathcal{L}_i^p(\rho),$$

and hence the corresponding channel is $\Lambda_i^p \approx (\mathbb{I} + \epsilon\mathcal{L}_i^p)$. The Choi matrix corresponding to the operation is given by $\mathcal{C}_p^z = \mathbb{I} \otimes \Lambda_i^p(|\psi_1\rangle\langle\psi_1|)$.

Let us now note that $\mathrm{Tr}\left[|\psi_2\rangle\langle\psi_2|\,\mathcal{C}_p^z\right] = \gamma_z(t)\epsilon$. For Markovian dynamics, the value of $\gamma_z(t)$ i.e decoherence rate is always non-negative. Therefore for the Markovian dynamics, the expectation value is non-negative. If the dynamics is non-Markovian, then the value of $\gamma_z(t)$ is negative and hence it gives a negative value to the corresponding expectation. Thus we can say that the Bell state $|\psi_2\rangle\langle\psi_2|$ serves as the non-Markovianity witness for σ_z dephasing operation.

Similarly for a dephasing channel in σ_x basis , we have $\frac{d\rho}{dt} = \mathcal{L}_x(\rho) = \gamma_x(t)(\sigma_x\rho\sigma_x - \rho)$. In this case, we observe that $\mathrm{Tr}\left[|\psi_3\rangle\langle\psi_3|\,\mathcal{C}_x\right] = \gamma_x(t)\epsilon < 0$ only when the dynamics is non-Markovian. Here $\mathcal{C}_x$ stands for the Choi matrix corresponding to the σ_x dephasing operation. Therefore the Bell state $|\psi_3\rangle\langle\psi_3|$ serves as the non-Markovianity witness for σ_x dephasing operation

In a similar way, for the dephasing operation in σ_y basis, it can be verified that the Bell state $|\psi_4\rangle\langle\psi_4|$ serves as the non-Markovianity witness.

Qubit depolarizing channel: The Lindblad equation of Pauli depolarizing channel is given by,

$$\frac{d\rho}{dt} = \sum_i \gamma_i(\sigma_i \rho \sigma_i - \rho),$$

where $i = x, y, z$ Let $\mathcal{C}_{dep}$ be the Choi matrix to the Pauli depolarizing operation. Then we note that

$$\begin{aligned}&\mathrm{Tr}\left[\left|\psi_4\right\rangle \left\langle\psi_4\right| \mathcal{C}^p_{dep}\right] = \gamma_y(t)\epsilon,\ \mathrm{Tr}\left[\left|\psi_3\right\rangle \left\langle\psi_3\right| \mathcal{C}^p_{dep}\right] = \gamma_x(t)\epsilon,\\ &\text{and } \mathrm{Tr}\left[\left|\psi_2\right\rangle \left\langle\psi_2\right| \mathcal{C}^p_{dep}\right] = \gamma_z(t)\epsilon.\end{aligned}$$

It is clear that the dynamics is non-Markovian if any one or more of the above expectation values is negative. As mentioned earlier, this kind of non-Markovianity includes "eternal" non-Markovianity in literature. Therefore this protocol provides an operational way to detect "eternal" non-markovianity via Bell measurement.

Similarly, any other combination of these three channels, Bell measurements can detect non-Markovianity.

Amplitude damping channel: The Lindblad equation of amplitude damping channel is given by,

$$\frac{d\rho}{dt} = \gamma_{amp}(t)(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}),$$

where, $\sigma_+ = \left|1\right\rangle \left\langle 0\right| \ \ and \ \ \sigma_- = \left|0\right\rangle \left\langle 1\right|$. Let $\mathcal{C}_{amp}$ be the Choi matrix corresponding to the amplitude damping channel. Note that $\mathrm{Tr}\left[\left|\psi_{3,4}\right\rangle \left\langle\psi_{3,4}\right| \mathcal{C}_{amp}\right] = \frac{\gamma_{amp}(t)}{4}\epsilon < 0$, when the dynamics is non-Markovian. Therefore the Bell states $\left|\psi_{3,4}\right\rangle \left\langle\psi_{3,4}\right|$ serves as non-Markovianity witness for amplitude damping channels.

Thermal channel: The Lindblad equation of thermal channel is given by,

$$\frac{d\rho}{dt} = \gamma(t)(n+1)(\sigma_-\rho\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-, \rho\}) \\ + \gamma(t)n(\sigma_+\rho\sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+, \rho\}),$$

where $n = \frac{1}{\exp(\frac{E}{KT})-1}$, (with E being the two level energy gap) is known as Plank number. We note that $\mathrm{Tr}\left[|\psi_{3,4}\rangle\langle\psi_{3,4}|\,\mathcal{C}_{th}\right] = (2n+1)\frac{\gamma(t)}{4}\epsilon < 0$ when the dynamics is non-Markovian. Here $\mathcal{C}_{th}$ stands for the corresponding Choi matrix. Therefore $|\psi\rangle_{3,4}\langle\psi|_{3,4}$ serves as the non-Markovianity witness for thermal channels.

Hence, given any unknown qubit dynamics, if it falls under the category of dephasing, depolarising or amplitude damping or thermal operations, one can detect its non-Markovianity without knowing the operation itself. Given the dynamics, one has to apply it on one part of the maximally entangled state $|\psi_1\rangle\langle\psi_1|$ and then the expectation values with respect to four maximally entangled Bell states have to be calculated. If the expectation values come out to be non-negative for all value of t and ϵ, the dynamics is Markovian. On contrary the negative expectation value shows the signature of non-Markovianity.

Thus we have shown in this section that for a substantial number of different qubit channels, Bell measurement can reveal their non-Markovianity without any prior knowledge of the channels completely. This provides us a huge operational advantage for detecting non-Markovianity of qubit dynamics.

5.3 Alternative construction of non-Markovianity witness:

As discussed above, the construction of witness depends on the eigenvalues of the corresponding Choi operator. For a large dimensional system, computing the eigenvalues and the corresponding projectors may be difficult. Therefore we adopt an alternative formalism [168] based on the structure of $\mathbb{F}_{\mathcal{M}}^{\epsilon}$.

To construct the witness, we first need to prove the existence of nearest Markovian Choi operator corresponding to some non-Markovian Choi operator. Consider $\mathcal{N}$ to be a non-Markovian operation having Choi operator $\mathcal{C}^{\mathcal{N}}(t)$. The distance between $\mathcal{C}^{\mathcal{N}}(t)$ and $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ is given by $M = \inf_{\mathcal{C}^{\mathcal{M}}(t)\in\mathbb{F}_{\mathcal{M}}^{\epsilon}} D(\mathcal{C}^{\mathcal{N}}(t) \mid \mathcal{C}^{\mathcal{M}}(t))$, where $D(\cdot \mid \cdot)$ is any proper metric. We now prove the following theorem.

Theorem 9: Corresponding to any non-Markovian Choi operator, there always exists a nearest Markovian Choi operator.

Proof: Fixing a non-Markovian Choi operator $\mathcal{C}^{\mathcal{N}}(t)$, we define a function $g : \mathbb{F}_{\mathcal{M}}^{\epsilon} \to \mathbb{R}$ by setting, $g(\mathcal{C}^{\mathcal{M}}(t))=D(\mathcal{C}^{\mathcal{N}}(t) \mid \mathcal{C}^{\mathcal{M}}(t))\ \forall\ \mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}_{\mathcal{M}}^{\epsilon}$. Clearly g is a continuous function on the set $\mathbb{F}_{\mathcal{M}}^{\epsilon}$. Moreover since $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ is compact, $\exists\ \mathcal{C}^{\mathcal{M}_*}(t) \in \mathbb{F}_{\mathcal{M}}^{\epsilon}$ such that $g(\mathcal{C}^{\mathcal{M}_*}(t))=\inf_{\mathcal{C}^{\mathcal{M}}(t)\in\mathbb{F}_{\mathcal{M}}^{\epsilon}} g(\mathcal{C}^{\mathcal{M}}(t))$. Hence infimum is achieved by some Markovian Choi operator. $\square$

Having proved the existence of nearest Markovian Choi operator, now it will be interesting to ask whether or under which condition the nearest Markovian Choi operator corresponding to a non-Markovian Choi operator is unique. Thus follows the next proposition.

Proposition 9: Let $\mathcal{C}^{\mathcal{N}}$ be a non-Markovian Choi operator. Then $\mathcal{C}^{\mathcal{M}_*}(t)$ is the unique nearest Markovian Choi operator if and only if for all $\mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}_{\mathcal{M}}^{\epsilon}$, $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}'}(t))(\mathcal{C}^{\mathcal{M}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))] \leq 0$.

Proof: To prove the sufficient part, let for all Markovian Choi operators $\mathcal{C}^{\mathcal{M}}(t)$, $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{M}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))] \leq 0$. Considering $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))^2]$, and by adding and subtracting $\mathcal{C}^{\mathcal{M}_*}$ we have, $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))^2] - \mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))^2] \geq -2\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{M}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))]$. Using the hypothesis we have $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))^2] \geq \mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))^2]$. This shows $\mathcal{C}^{\mathcal{M}_*}(t)$ is the nearest Markovian Choi operator corresponding to the non-Markovian Choi operator $\mathcal{C}^{\mathcal{N}}$.

To prove the necessary part, let $\mathcal{C}^{\mathcal{M}_*}(t)$ be the nearest Markovian Choi operator corresponding to non- Markovian Choi operator $\mathcal{C}^{\mathcal{N}}(t)$. Then, $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))^2] \geq \mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))^2]$ for any Markovian Choi operator $\mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}_{\mathcal{M}}^{\epsilon}$. This implies $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{M}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))] \leq \dfrac{1}{2}\ \mathrm{Tr}[(\mathcal{C}^{\mathcal{M}_*}(t) - \mathcal{C}^{\mathcal{M}}(t))^2]$. Since

$\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is convex, let $\mathcal{C}^{\mathcal{M}}(t) = (1-\mu)\mathcal{C}^{\mathcal{Q}}(t) + \mu\mathcal{C}^{\mathcal{M}_*}(t)$ with $0 < \lambda < 1$, where $\mathcal{C}^{\mathcal{Q}}(t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}$. Therefore $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))((1-\mu)\mathcal{C}^{\mathcal{Q}}(t) + \mu\mathcal{C}^{\mathcal{M}_*}(t) - \mathcal{C}^{\mathcal{M}_*}(t))] \leq \frac{1}{2}$ $\mathrm{Tr}[((1-\mu)\mathcal{C}^{\mathcal{Q}}(t) + \mu\mathcal{C}^{\mathcal{M}_*}(t) - \mathcal{C}^{\mathcal{M}_*}(t))^2]$, which gives the inequality $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{Q}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))] \leqslant \frac{1}{2}(1-\mu)\mathrm{Tr}[(\mathcal{C}^{\mathcal{M}_*}(t) - \mathcal{C}^{\mathcal{Q}}(t))^2]$. Letting $\mu \to 1$, we have the result.

To prove the uniqueness of $\mathcal{C}^{\mathcal{M}_*}(t)$, let $\mathcal{C}^{\mathcal{M}^\infty}(t)$ be another Markovian Choi operator which minimizes $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}}(t))^2]$. Then $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{M}^\infty}(t) - \mathcal{C}^{\mathcal{M}_*}(t))] \leqslant 0$ and $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}^\infty}(t))(\mathcal{C}^{\mathcal{M}_*}(t) - \mathcal{C}^{\mathcal{M}^\infty}(t))] \leqslant 0$ together implies $\mathcal{C}^{\mathcal{M}_*}(t) = \mathcal{C}^{\mathcal{M}^\infty}(t)$, proving the uniqueness of the nearest Markovian Choi operator. $\square$

Armed with this proposition, we now prove the following theorem.

Theorem 10: Let $\mathcal{C}^{\mathcal{M}_*}(t)$ be the nearest Markovian Choi operator to a non-Markovian Choi operator $\mathcal{C}^{\mathcal{N}}$. Then

$$\mathcal{W} = c_0\mathbb{I} + \mathcal{C}^{\mathcal{M}_*}(t) - \mathcal{C}^{\mathcal{N}}(t), \tag{5.2}$$

with $c_0 = \mathrm{Tr}(\mathcal{C}^{\mathcal{M}_*}(t)(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t)))$ is a non- Markovianity witness for $\mathcal{C}^{\mathcal{N}}(t)$.

Proof: We verify that, $\mathrm{Tr}[\mathcal{W}\mathcal{C}^{\mathcal{S}}(t)] = -\mathrm{Tr}[(\mathcal{C}^{\mathcal{S}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))]$ for any Choi operator $\mathcal{C}^{\mathcal{S}}(t)$. To prove $\mathcal{W}$ is a non-Markovianity witness for $\mathcal{C}^{\mathcal{N}}(t)$ it is enough to show 1) $\mathcal{W}$ is Hermitian, 2) $\mathrm{Tr}[\mathcal{W}\mathcal{C}^{\mathcal{M}}(t)] \geqslant 0$ $\forall$ $\mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}$ and $\mathrm{Tr}[\mathcal{W}\mathcal{C}^{\mathcal{N}}(t)] < 0$.

$\mathcal{W}$ is Hermitian according to its definition. Since $\mathrm{Tr}[\mathcal{W}\mathcal{C}^{\mathcal{S}}(t)] = -\mathrm{Tr}[(\mathcal{C}^{\mathcal{S}}(t)$-$\mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))]$ holds for any Choi operator, it also holds for any $\mathcal{C}^{\mathcal{M}}(t)$ in $\mathbb{F}^{\epsilon}_{\mathcal{M}}$.

Since $\mathcal{C}^{\mathcal{M}_*}(t)$ is the nearest Markovian Choi operator for $\mathcal{C}^{\mathcal{N}}(t)$, we have $\mathrm{Tr}[(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))(\mathcal{C}^{\mathcal{M}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))] \leqslant 0$. Hence it follows that

$$\mathrm{Tr}[\mathcal{W}\mathcal{C}^{\mathcal{M}}(t)] \geqslant 0 \ \ \forall \ \ \mathcal{C}^{\mathcal{M}}(t) \in \mathbb{F}^{\epsilon}_{\mathcal{M}}.$$

Using the trace preservation property of quantum operations, we get

$$\mathrm{Tr}[\mathcal{W}\mathcal{C}^{\mathcal{N}}(t)] = -\mathrm{Tr}(\mathcal{C}^{\mathcal{N}}(t) - \mathcal{C}^{\mathcal{M}_*}(t))^2 < 0$$

□

So far we have constructed the theory of linear witnesses to detect non-Markovian dynamics, we ask the immediate following question that, whether linear witnesses are sufficient to determine all the non-Markovian Choi operators. In the theory of entanglement detection, we know that non-linear improvement of witnesses gives us further advantages to detect entanglement [169–171]. In the following, we discuss such possibilities for non-Markovianity detection.

5.4 Existence of non-linear witnesses:

After constructing the structure of linear witnesses to detect the non-Markovian Choi operators, a very legitimate question should be the following. How many linear witnesses are enough to capture all non- Markovian Choi operators. To answer this question, we need to investigate the geometry of $\mathbb{F}^{\epsilon}_{\mathcal{M}}$; *i.e.* precisely whether the set $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ forms a polytope determined by intersection of finitely many half-spaces obtained from linear witnesses. In analogy to Ref. [172], we surmise that these finitely many witnesses are tangents to $\mathbb{F}^{\epsilon}_{\mathcal{M}}$. Minkowski's theorem [173] states that every polytope in $\mathbb{R}^n$ is the convex hull of finitely many extreme points. Therefore, if there exists finitely many extreme points for a given convex and compact set, finitely many linear witnesses will suffice for separating all the entities outside that given set. The task is now to determine the number of extreme points of the set $\mathbb{F}^{\epsilon}_{\mathcal{M}}$. We resolve this issue by proving the following theorem.

Theorem 11: The convex compact set of all Markovian Choi operators do not form a polytope.

proof: In a convex set extreme points are those members of the set which have no non-trivial decompositions in terms of convex combinations of other points of the set. Since the Markovian Choi operators are always valid physical states, the pure states, if there any, will lie on the vertices of the set. Hence, to prove the theorem, it is enough to show that there exists uncountable many pure Choi states. Consider the set of all Unitary channels: $\{\mathcal{U}_\alpha(t_2, t_1) \mid \alpha \in \mathrm{J}\}$, where J is an uncountable index set. For each α , $\mathcal{U}_\alpha$ is a superoperator whose action is given by $\mathcal{U}_\alpha(\rho) = U_\alpha \rho U_\alpha^\dagger$, with U_α being unitary operators. Unitary operations are divisible, *i.e.* $\mathcal{U}_\alpha(t_3, t_1) = \mathcal{U}_\alpha(t_3, t_2) \circ \mathcal{U}_\alpha(t_2, t_1)$, with $t_3 \geq t_2 \geq t_1$, $\forall\, t_1, t_2, t_3$. For a more detailed description, we consider unitary evolution $\mathcal{U}_\alpha$ with unitary operator U_α and time independent Hamiltonians H_α, as $U_\alpha(t_2, t_1) = \exp\left(-i(t_2 - t_1)H_\alpha\right)$. Under small time interval approximation, $U_\alpha(t_1 + \epsilon, t_1) \approx \mathbb{I} + \epsilon H_\alpha$.

It is straight forward to check that the evolution $\rho(t_1+\epsilon) = U_\alpha(t_1+\epsilon, t_1)\rho(t_1)U_\alpha^\dagger(t_1+\epsilon, t_1)$ is divisible. Thus, the Choi operator corresponding to some Unitary operation $\mathcal{U}_{\alpha_0}(t_2, t_1)$, given by $\mathcal{C}_{\alpha_0}^{\mathcal{U}} = \mathbb{I} \otimes \mathcal{U}_{\alpha_0}(t + \epsilon, t)(|\phi\rangle\langle\phi|)$, is a pure maximally entangled state and hence it is an extreme point of $\mathbb{F}_{\mathcal{M}}^{\epsilon}$. Since $\mathcal{I}$ is uncountable, there exists uncountably many such pure maximally entangled states for given dimensions. Therefore there are uncountably many extreme points of $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ and hence it does not form a polytope. □

As a consequence of **Theorem 11**, we surmise the importance of non-linear witnesses, to improve upon the efficiency of its linear counterpart. Since $\mathbb{F}_{\mathcal{M}}^{\epsilon}$ does not form a polytope, finite number of hyperplanes will not be sufficient to detect all the non-Markovian Choi operators. Therefore non-linear improvement of witnesses are necessary to detect them. In Fig. 5.2 we schematically justify the necessity of non-linear witnesses.

From the above discussion, we have found that some of the pure maximally entangled states are in the set of all extreme points $Ext(\mathbb{F}_{\mathcal{M}}^{\epsilon})$ of the set $\mathbb{F}_{\mathcal{M}}^{\epsilon}$. We know that they are also among the extreme points of the state space $\mathbb{S}$. This fact tells us that $Ext\,(\mathbb{S}) \cap Ext(\mathbb{F}_{\mathcal{M}}^{\epsilon})$ is non-empty. But it is also evident that not all the pure maximally entangled states are in $Ext(\mathbb{F}_{\mathcal{M}}^{\epsilon})$. This is because of the fact that the map is locally applied on one side of a bipartite maximally entangled state to construct the Choi states. It is therefore clear that the maximally entangled states generated by applying local unitaries on the other side, will not be among the set of Choi states. Those states, though among the extreme points of $\mathbb{S}$, will not be in $\mathbb{F}_{\mathcal{M}}^{\epsilon}$. The most obvious open question is then whether $Ext(\mathbb{F}_{\mathcal{M}}^{\epsilon})$ is a

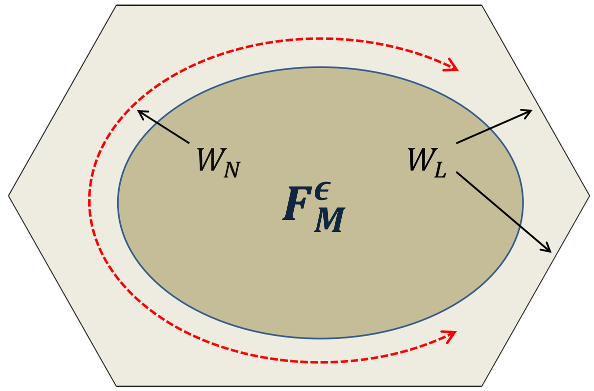

Figure 5.2: (Colour online) The figure depicts a cross section of the set $\mathbb{F}^{\epsilon}_{\mathcal{M}}$. The outer polygon represents a convex set with a polytope structure, whose sides are optimal linear witnesses (W_L). Clearly we understand that if the set had a polytope structure, finite linear witnesses would have been sufficient to distinguish all the points not in the set. Since $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ does not have that, as depicted by inner ellipse, non-linear improvement is always possible. The red dashed line indicates such a non-linear witness W_N.

strict subset of $Ext\,(\mathbb{S})$. However, we make the conjecture that this is not the case. The argument behind this statement is the following. Since $\mathbb{F}^{\epsilon}_{\mathcal{M}} \subset \mathbb{S}$, there are valid physical states not contained in $\mathbb{F}^{\epsilon}_{\mathcal{M}}$. We have already proved that there exists pure states not contained in $Ext(\mathbb{F}^{\epsilon}_{\mathcal{M}})$. Therefore, there can be mixed Choi states having no non-trivial state decomposition in terms of the pure states in $Ext(\mathbb{F}^{\epsilon}_{\mathcal{M}})$. Though they always have the same in terms of pure states which are in $Ext(\mathbb{S})$.

It can be shown that the set of all Choi states $\mathbb{C}_{\mathcal{A}}$ is also a convex set under the small time interval approximation. $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is of course a strict subset of $\mathbb{C}_{\mathcal{A}}$. We have shown earlier that $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ is also a strict subset of $\mathbb{S}$. The set $\mathbb{F}^{\epsilon}_{\mathcal{N}} = \mathbb{C}_{\mathcal{A}} \setminus \mathbb{F}^{\epsilon}_{\mathcal{M}}$ contains all the non-Markovian Choi operators. Clearly all the elements of $\mathbb{F}^{\epsilon}_{\mathcal{N}}$ are not valid quantum states, because non-Markovian operations breaks CP-divisibility. Therefore, it is evident that $\mathbb{F}^{\epsilon}_{\mathcal{M}} = \mathbb{S} \cap \mathbb{C}_{\mathcal{A}}$. In the following Fig. 5.2, we depict this discussion schematically.

To summarize the chapter, we have exploited the convex and compact structure of CP divisible *i.e.* Markovian operations under small time interval approximation

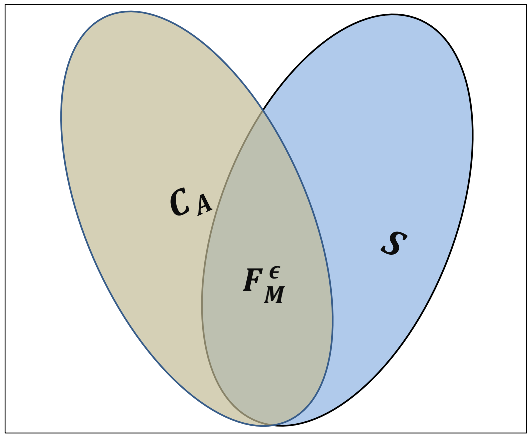

Figure 5.3: (Colour online) Here we represent the convex sets $\mathbb{S}$ and $\mathbb{C}_{\mathcal{A}}$. Their intersection is another convex set $\mathbb{F}^{\epsilon}_{\mathcal{M}}$. We can also understand from the diagram, that some of the extreme points of $\mathbb{F}^{\epsilon}_{\mathcal{M}}$ are mixed density matrices.

to prove the existence of linear witness of non-Markovianity. We take the help of geometric Hahn-Banach separation theorem in this context. We have constructed linear witness of non-Markovianity using various techniques. Interestingly two qubit maximally entangled states can be used to detect qubit non-Markovian dynamics. We further explore the convex structure of Markovian operations. We have shown that they do not form a polytope. This proves the necessity to consider non-linear improvement of non-Markovianity witness. So far we have discussed about the convex resource theory of non-Markovianity and its detection using the idea of witness. Now we would like to study non-Markovianity as a resource in the backdrop of other resource theories. We move on to the next chapter to address this point.

Chapter 6

Thermodynamic utility of Non-Markovianity from the perspective of resource interconversion

Quantum information theory deals with interconversion between various quantum resources. We have established the framework of convex resource theory of non-Markovianity and the idea of non-Markovianity detection in the previous chapters. In this chapter which is based on [174], we would like to address how non-Markovianity in terms of information backflow gets converted into other resources. Precisely we want to study the resource theory of non-Markovianity in the backdrop of other resource theories. We shall discuss from the perspective of resource theory of purity and resource theory of thermodynamics. Both of them are well established resource theories. In the resource theory of purity unital operations are free operations. In the resource theory of thermodynamics the same role is played by the thermal operations. We shall try to characterize their structures. Entropy of the system will play an important role in the whole discussion. The concept of entropy production rate is well studied in thermodynamics. In this chapter we shall formulate an idea of generalised entropy production rate and discuss its connection with non-Markovian backflow of information. In the next section we start with the discussion on entropy production rate.

6.1 Entropy Production Rate

Entropy production rate (EPR) is defined as the negative time derivative of the relative entropy between the instantaneous state and the thermal state [175]:

$$\sigma(t) = -\frac{d}{dt}S(\rho(t)||\tau_\beta), \tag{6.1}$$

where $S(A||B) = \mathrm{Tr}[A(\ln A - \ln B)]$ is the von-Neumann relative entropy and $\tau_\beta = e^{-\beta H}/\mathcal{Z}$ is the thermal state of the system at inverse temperature β, where $\mathcal{Z} = \mathrm{Tr}[e^{-\beta H}]$ is the partition function. Under thermal operation, the thermal state is the only fixed point of the dynamics. It can be shown that $\frac{d}{dt}S(\rho(t)||\tau_\beta) = \mathrm{Tr}[\dot{\rho}(t)\ln\rho(t)] + \beta\mathrm{Tr}[H\dot{\rho}(t)]$, giving rise to the relation

$$\sigma(t) = \frac{d}{dt}S(t) - \beta\mathcal{J} = \frac{d}{dt}\left(S(t) - \beta\mathcal{W}(t)\right), \tag{6.2}$$

where $S(t) = -\mathrm{Tr}[\rho(t)\ln\rho(t)]$ is the von-Neumann entropy and $\mathcal{J} = \mathrm{Tr}[H\dot{\rho}(t)]$ is the heat current. $\mathcal{W}(t) = \mathrm{Tr}[H(\rho(t) - \tau_\beta)]$ is the maximum work that can be extracted from the system by applying thermal operation. EPR can also be generalized for Réyni divergence as $\sigma^\gamma(t) = -\frac{d}{dt}S_\gamma(\rho(t)||\tau_\beta)$, where $S_\gamma(\rho(t)||\tau_\beta) = \frac{1}{\gamma-1}\ln\left(\mathrm{Tr}[\rho(t)^\gamma\tau_\beta^{1-\gamma}]\right)$ $(\gamma > 0)$, is the Réyni relative entropy. We show that under unital operation and thermal operation, $\sigma^\gamma(t)$ is positive under divisible CPTP evolution.

6.1.1 Non-Markovian backflow of information under unital operations

We first consider the backdrop of resource theory of purity [134, 176]. A successful approach to the fundamental aspects of thermodynamics can be adopted, by considering purity as a resource. This can be done from different operational perspectives, depending on the set of of easily implementable free operations. One convincing approach towards this end is to consider noisy or unital operations as the free operations. An operation is said to be unital if the identity operator is the only fixed point of the operation. In the following theorem we analyse the characteristics of Lindblad operators [117] for unital operations.

Theorem 12: For all unital dynamical maps having corresponding Lindblad generators, the Lindblad operators are normal.

Proof. : We first prove the sufficiency condition, *viz.*, if the Lindbladians are normal then the dynamics is unital. Let us consider a dynamical map: $\rho(t) = \Lambda(\rho(0))$ having a Lindbladian generator

$$\dot{\rho}(t) = \mathcal{L}(\rho(t)) = \sum_{\alpha=1}^{n \leq d_S^2} \Gamma_\alpha(t) \left(A_\alpha \rho(t) A_\alpha^\dagger - \frac{1}{2} A_\alpha^\dagger A_\alpha \rho(t) - \frac{1}{2} \rho(t) A_\alpha^\dagger A_\alpha \right)$$

.

To show the unitality of the map it is enough to show that $\mathcal{L}(\mathbf{I}_S) = 0$, where $\mathbf{I}_S$ stands for identity matrix corresponding to the system dimension. Putting $\rho(t) = \mathbf{I}_S$ in the Lindblad master equation we have,

$$\mathcal{L}(\mathbf{I}_S) = \sum_{\alpha=1}^{n \leq d_S^2} \Gamma_\alpha(t) \left(A_\alpha A_\alpha^\dagger - \frac{1}{2} A_\alpha^\dagger A_\alpha - \frac{1}{2} A_\alpha^\dagger A_\alpha \right)$$

Using the property of normality $A_\alpha A_\alpha^\dagger = A_\alpha^\dagger A_\alpha \;\; \forall \alpha$, we have $\mathcal{L}(\mathbf{I}_S) = 0$.

We now prove the necessity condition, *viz.*, if the dynamics is unital then the Lindblad operators are normal. Let us now consider $A_\alpha = \sum_{ij\alpha} C_{ij}^\alpha \left| i \right\rangle \left\langle j \right|$, where

$|i\rangle$, $|j\rangle$ forms a complete set of orthonormal basis vectors. Now,

$$
\begin{aligned}
\mathcal{L}(\mathbb{I}_S) &= \textstyle\sum_\alpha \mathit{\Gamma}_\alpha(t)(A_\alpha A_\alpha^\dagger - A_\alpha^\dagger A_\alpha), \\
&= \textstyle\sum_{ijkl\alpha} \mathit{\Gamma}_\alpha(t) \left(C_{ij}^\alpha |i\rangle \langle j| (C_{kl}^\alpha)^* |1\rangle \langle k| - (C_{il}^\alpha)^* |1\rangle \langle k| C_{ij}^\alpha |i\rangle \langle j|\right), \\
&= \textstyle\sum_{ijk\alpha} \mathit{\Gamma}_\alpha(t) C_{ij}^\alpha (C_{kj}^\alpha)^* |i\rangle \langle k| - \sum_{ijl\alpha} \mathit{\Gamma}_\alpha(t) C_{ij}^\alpha (C_{il}^\alpha)^* |1\rangle \langle j|, \\
&= \textstyle\sum_{ik} \left(\sum_{j\alpha} \mathit{\Gamma}_\alpha(t) C_{ij}^\alpha (C_{kj}^\alpha)^*\right) |i\rangle \langle k| - \sum_{lj} \left(\sum_{i\alpha} \mathit{\Gamma}_\alpha(t) C_{ij}^\alpha (C_{il}^\alpha)^*\right) |1\rangle \langle j|, \\
&= \textstyle\sum_{ik} \mathit{\Lambda}_{ik}(t) |i\rangle \langle k| - \sum_{lj} \mathit{\Lambda}_{jl}(t) |1\rangle \langle j|, \\
&= \textstyle\sum_{mn} \mathit{\Lambda}_{mn}(t) |m\rangle \langle n| - \sum_{mn} \mathit{\Lambda}_{nm}(t) |m\rangle \langle n|, \\
&= \textstyle\sum_{mn} (\mathit{\Lambda}_{mn}(t) - \mathit{\Lambda}_{nm}(t)) |m\rangle \langle n|,
\end{aligned}
$$

Therefore, $\mathcal{L}(\mathbf{I}_S) = 0$ implies $\mathit{\Lambda}_{mn} = \mathit{\Lambda}_{nm}\ \forall n,\ m$. Consider $A_{jk} = |j\rangle \langle k|$. The Lindblad evolution for the unital channel can be expressed as

$$
\begin{aligned}
\mathcal{L}(\rho(t)) = \tfrac{1}{2} \textstyle\sum_{j,k} \mathit{\Lambda}_{jk}(t) &\left(A_{jk}\rho(t)A_{jk}^\dagger - \tfrac{1}{2}A_{jk}^\dagger A_{jk}\rho(t) - \tfrac{1}{2}\rho(t)A_{jk}^\dagger A_{jk}\right) \\
+ \tfrac{1}{2} \textstyle\sum_{j,k} \mathit{\Lambda}_{kj}(t) &\left(A_{jk}^\dagger \rho(t) A_{jk} - \tfrac{1}{2}A_{jk}A_{jk}^\dagger \rho(t) - \tfrac{1}{2}\rho(t)A_{jk}A_{jk}^\dagger\right),
\end{aligned}
$$

since $A_{jk}^\dagger = A_{kj}$. Using $\mathit{\Lambda}_{mn} = \mathit{\Lambda}_{nm}\ \forall n,\ m$, this equation can be modified to

$$
\begin{aligned}
&\mathcal{L}(\rho(t)) = \tfrac{1}{2} \textstyle\sum_{jk} \mathit{\Lambda}_{jk}(t) \left(H_{jk}\rho(t)H_{jk}^\dagger - \tfrac{1}{2}H_{jk}^\dagger H_\alpha \rho(t) - \tfrac{1}{2}\rho(t)H_{jk}^\dagger H_{jk}\right) \\
&+\tfrac{1}{2} \textstyle\sum_{jk} \mathit{\Lambda}_{jk}(t) \left(\bar{H}_{jk}\rho(t)\bar{H}_{jk}^\dagger - \tfrac{1}{2}\bar{H}_{jk}^\dagger \bar{H}_{jk}\rho(t) - \tfrac{1}{2}\rho(t)\bar{H}_{jk}^\dagger \bar{H}_{jk}\right),
\end{aligned}
$$

where $H_{jk} = \frac{A_{jk}+A_{jk}^\dagger}{\sqrt{2}}$ and $\bar{H}_{jk} = \frac{i(A_{jk}-A_{jk}^\dagger)}{\sqrt{2}}$ are Hermitian operators which are normal. □

This theorem completely characterizes the Lindbladians for unital operations. Using the theorem, we prove the following corollary.

Corollary: For unital quantum dynamical processes, non-Markovianity is necessary to drive the system away from equilibrium.

Proof. : A unital operation can always be represented by the dynamical map:

$$
\mathit{\Lambda}_U(\rho(0)) = \mathrm{Tr}_B \left[V_{SB}\left(\rho(0) \otimes \frac{\mathbf{I_B}}{d_B}\right) V_{SB}^\dagger\right],
$$

where V_{SB} is a global unitary process acting over the total system-environment state, $\mathbf{I}_B$ is the identity matrix for the environment B, and d_B is the dimension of the environment. For the mentioned operation, $\mathbf{I}_S/d_S$ is the fixed point, which corresponds to the thermal state at infinite temperature ($\beta \to 0$). Here d_S is the dimension of the system. For unital evolution, we use the identity $S(\rho||\frac{\mathbf{I}}{d_S}) = \ln d_S - S(\rho)$, to show that the entropy production rate can be defined as $\sigma_U(t) = \frac{d}{dt}S(t)$.

The result holds for generalized Réyni entropy also. In a previous work [177], it has been shown that, for the Lindblad type evolution of generalized Réyni entropy $S_\gamma(t) = \frac{1}{1-\gamma}\ln \mathrm{Tr}[\rho^\gamma(t)],\ (\gamma > 0)$, we have

$$\frac{d}{dt}S_\gamma(t) = 2\sum_\alpha \mathit{\Gamma}_\alpha^U(t)\chi_\alpha(t),$$

where

$$\chi_\alpha(t) = \frac{\gamma}{1-\gamma}\frac{1}{\mathrm{Tr}[\rho^\gamma(t)]}\mathrm{Tr}\left(\rho^{\gamma-1}(t)A_\alpha\rho(t)A_\alpha^\dagger - \rho^\gamma(t)A_\alpha^\dagger A_\alpha\right).$$

It has been proved previously [177] that

$$\chi(t) > \langle[A_\alpha^\dagger, A_\alpha]\rangle_\gamma,$$

where $\langle X\rangle_\gamma = \frac{\mathrm{Tr}[X\rho^\gamma(t)]}{\mathrm{Tr}[\rho^\gamma(t)]}$. Since for unital evolutions, A_αs are normal, we always have $\chi(t) > 0$. Therefore the generalized EPR $\sigma_U^\gamma(t) \geq 0$, for all divisible evolutions ($\mathit{\Gamma}_\alpha^U(t) \geq 0 \quad \forall\alpha$). Therefore $\sigma_U^\gamma(t)$ can only be negative when divisibility of the dynamical process breaks down ($\mathit{\Gamma}_\alpha^U(t) \leq 0$). This proves that non-Markovianity is necessary to drive the system away from equilibrium. □

Following **Corollary** , we present an important result connecting non-Markovian information backflow and resource theory of purity.

Result 1: The rate of change of purity under a unital dynamical process can be represented as

$$\tfrac{dP}{dt} = -\textstyle\sum_\alpha \mathit{\Gamma}_\alpha^U(t)Q(A_\alpha), \tag{6.3}$$

where $\Gamma_\alpha^U(t)$ and A_α are respectively the Lindblad coefficients and Lindblad operators for unital evolution. The asymmetry of an operator with respect to a quantum state can be defined as $Q(O_i) = ||[\rho, O_i]||^2_{HS}$, where $||.||_{HS}$ denotes the Hilbert-Schmidt norm.

Proof. : The purity of a state is given as: $P(t) = \mathrm{Tr}[\rho^2]$. Therefore we have

$$\frac{d}{dt}P(t) = 2\mathrm{Tr}[\rho(t)\frac{d\rho(t)}{dt}] = 2\mathrm{Tr}[\rho(t)\mathcal{L}(\rho(t))]$$

Using the the property of normal operator: $A_\alpha^\dagger A_\alpha = A_\alpha A_\alpha^\dagger$ and the cyclic property of trace, we get

$$\begin{aligned} Q(A_\alpha) &= \mathrm{Tr}\left[(\rho(t)A_\alpha - A_\alpha\rho(t))(A_\alpha^\dagger\rho(t) - \rho(t)A_\alpha^\dagger)\right], \\ &= -2\mathrm{Tr}\left[\rho(t)\left(A_\alpha\rho(t)A_\alpha^\dagger - \tfrac{1}{2}\{\rho(t), A_\alpha^\dagger A_\alpha\}\right)\right]. \end{aligned}$$

Thus, using the form of $\mathcal{L}_U(\rho(t))$, we get

$$\frac{d}{dt}P(t) = -\sum_\alpha \Gamma_\alpha^U(t)Q(A_\alpha).$$

□

Therefore, from **Result 1**, it is evident that purity can only regenerated in the non-Markovian region ($\Gamma_\alpha^U(t) < 0$) of the dynamics. It shows that in the backdrop of resource theory of purity, non-Markovianity can be converted into purity, via information backflow.

6.1.2 Non-Markovian backflow under thermal operations

Let us now extend our study of classifying Lindblad dynamics for thermal operations. An operation Λ_{th} is said to be thermal if the thermal state of the system τ_β is the fixed point of the operation and the free energy is a monotonically decreasing function under the same operation. Mathematically, a thermal operation is described as

$$\Lambda_{th}(\rho(0)) = \mathrm{Tr}_B\left[\mathcal{V}_{SB}\left(\rho(0)\otimes\mu_\beta\right)\mathcal{V}_{SB}^\dagger\right], \tag{6.4}$$

where $\mathcal{V}_{SB}$ is a energy preserving unitary operation acting on the system-environment composite Hilbert space and μ_β is the thermal state of the environment at a given temperature β. The energy preservation condition invokes the following relation $[\mathcal{V}_{SB}, H_S + H_B] = 0$, where H_S and H_B are the system and environment Hamiltonian respectively. Under these restrictions over the allowed operations, it can be shown that the thermal state of the system τ_β is the fixed point of the dynamics, *i.e.* $\Lambda_{th}(\tau_\beta) = \tau_\beta$. Nevertheless, the Stinespring form [76,178] of thermal operation given in Eq. (6.4) is true for large class of unitary operators and it does not shed much light on the realistic constraints in experimental situations. Hence, our first goal in this subsection is the classification of the Lindblad generators corresponding to the thermal operations Λ_{th}. In this context it is important to note that full characterization of Lindblad generators corresponding to an arbitrary thermal operation is quite a difficult task since the class of energy preserving unitary operations is very large and there is no unique way to capture all of them. There have been attempts to bypass this problem by considering physically realizable dynamics, such as conditional thermal operations [179] or elementary thermal operations [180]. Here we identify a significant portion of thermal operations having clear experimental significance. In the following theorem, by proving a sufficient condition for a Lindblad operation to be thermal, we identify a particular subset of thermal operations.

Theorem 13: If the Lindblad operators are restricted to be of the form of rank 1 projector: $A_{ij} = |j\rangle\langle i|$, with $\{|i\rangle\}$ being the energy eigenbasis and $\Gamma_{ij}(t)$s the corresponding Lindblad coefficients, then the condition for the operation to be a thermal operation is the detailed balance condition:

$$\Gamma_{ji}(t) = \Gamma_{ij}(t)e^{-\beta(E_i-E_j)}$$

Proof. Let us consider $\mathcal{L}_{th}$ to be the Lindblad generator corresponding to the thermal operation Λ_{th}. Gibbs preservation condition gives $\Lambda_{th}(\tau_\beta) = \tau_\beta$. Therefore

$\dfrac{d\tau_\beta}{dt} = \mathcal{L}_{th}(\tau_\beta) = 0,$

yields $\mathcal{L}_{th}(\tau_\beta) = 0$.

Now, $\mathcal{L}_{th}(\rho)= \sum_{ijk} \Gamma_{ij}(t)(|j\rangle\langle i|\rho|i\rangle\langle j| - \frac{1}{2}|i\rangle\langle i||\rho - \frac{1}{2}\rho|i\rangle\langle i|)$. Therefore,

$$\Lambda_{th}(\tau_\beta) = \sum_{ijk} \Gamma_{ij}(t)(|j\rangle\langle i| e^{-\beta E_k}|k\rangle\langle k||i\rangle\langle j| - \frac{1}{2}|i\rangle\langle i||k\rangle\langle k| e^{-\beta E_k} - \frac{1}{2}e^{-\beta E_k}|k\rangle\langle k||i\rangle\langle i|)$$

$$= \sum_{ij}(\Gamma_{ij}(t)e^{-\beta E_i} - \Gamma_{ji}(t)e^{-\beta E_j})|j\rangle\langle j|$$

Hence, $\Lambda_{th}(\tau_\beta) = 0$ implies $\Gamma_{ji}(t) = e^{-\beta(E_i - E_j)}\Gamma_{ij}(t)$ □

As a relevant example, consider a simple Markovian model, with a qubit is weakly coupled to a thermal bosonic environment. In absence of any external driving, the qubit eventually thermally equilibrates with the environment. Under Born-Markov approximation, the master equation for this model is given by

$$\begin{aligned}\dot{\rho}(\tilde{t}) = \tfrac{i}{\hbar}[\rho(\tilde{t}), H_0] + \gamma(n+1)\left(\sigma_-\rho(\tilde{t})\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-, \rho(\tilde{t})\}\right) \\ + \gamma n\left(\sigma_+\rho(\tilde{t})\sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+, \rho(\tilde{t})\}\right),\end{aligned} \tag{6.5}$$

where $H_0 = \hbar\Omega_0|1\rangle\langle 1|$ is the Hamiltonian of the system, γ is a constant parameter and $n = 1/(\exp(\hbar\Omega_0/K\tilde{T}_m) - 1)$ is the Planck number. Here σ_+ and σ_- are respectively the raising and lowering operators of the two level system, with $|1\rangle$ being the excited state of the same. This operation represents a thermal operation on a two level system. The two Lindblad operators corresponding to the operation are rank one projectors and hence satisfy all the properties stated in **Theorem 13**. It is straightforward to check, that the thermal state of the qubit corresponding to the bath temperature $\hat{T}_m$ and the system Hamiltonian H_0 is the only fixed point of this dynamics, proving the operation to be thermal.

At this stage, it is interesting to compare our class of thermal operations having rank 1 projectors as Lindblad operators, with a physically implementable subclass of thermal operations, namely elementary thermal operations [180]. It has been previously shown [180], that elementary thermal operations are those which satisfy the following two criteria:, (i) the map involves only two energy levels of the system, and (ii) it satisfies the detailed balance condition. It is clear from **Theorem 13** that if only two particular energy levels (i, j) are involved in the Lindblad type evolution, the sub-class of thermal operations that we have considered is nothing but the class of elementary thermal operations. The consequence of this finding is important from experimental perspectives. Two-level population dynamics can generally be realized by elementary thermal operations involving a single mode bosonic bath, and interestingly, the Jaynes-Cummings model can reproduce them to a satisfactory extent [180]. Therefore, it is evident that the class of thermal operations we consider in **Theorem 13** encompasses a considerable number of elementary thermal operations which are physically realizable in experimental situations.

We now focus on non-Markovianity and its importance from the perspective of resource inter-conversion. A very relevant aspect of quantum information theory is the study of interconversion of different resources. Quantum non-Markovianity is one of the resources which can be studied from the perspective of quantum thermodynamics. Here it becomes important to observe how different thermodynamic quantities respond under the presence of non-Markovianity. In order to do so, let us consider the role of non-Markovianity in the resource theory of thermodynamics. Under thermal operations, the free energy $\mathcal{F}(t) = \langle H \rangle - S(t)/\beta$ is a monotone and we prove that it obeys the following relation with EPR.

$$\beta \frac{d}{dt}\mathcal{F}(t) = -\sigma(t). \tag{6.6}$$

The proof follows from the observation that, the free energy can be expressed as $\mathcal{F}(t) = \frac{1}{\beta}S(\rho(t)||\tau_\beta) - \frac{1}{\beta}\ln \mathcal{Z}$. Therefore under thermal operation, we have $\beta\frac{d}{dt}\mathcal{F}(t) = \frac{d}{dt}S(\rho(t)||\tau_\beta) = -\sigma(t)$.

A more general definition of free energy [181] can be stated as $\mathcal{F}^\gamma(t) = \frac{1}{\beta}S^\gamma(\rho(t)||\tau_\beta) - \frac{1}{\beta}\ln \mathcal{Z}$. Using this definition, (6.6) can be generalized for Réyni divergence as: $\beta\frac{d}{dt}\mathcal{F}^\gamma(t) = -\sigma^\gamma(t)$. This result shows that negative EPR is necessary and sufficient for free energy backflow.

Corollary : Under thermal operations Λ_{th}, which is CP-divisible, EPR is always positive.

Proof. : To prove the above corollary, we use the following two facts.

1. Monotonicity of relative entropy under CPTP maps: $S(\Lambda(\rho)||\Lambda(\sigma)) \leq S(\rho||\sigma)$.

2. Thermal state τ_β is a fixed point under thermal operation: $\Lambda_{th}(\tau_\beta) = \tau_\beta$.

We have

$$\begin{aligned} S(\rho(t)||\tau_\beta) &\geq S(\Lambda_{th}(t+\delta,t)(\rho(t))||\Lambda_{th}(t+\delta)(\tau_\beta)) \\ &= S(\Lambda_{th}(t+\delta,t)(\rho(t))||\tau_\beta) \end{aligned}$$

Therefore, we have

$$\tfrac{d}{dt}S(\rho(t)||\tau_\beta) = \lim_{\delta\to 0} \tfrac{S(\rho(t+\delta)||\tau_\beta)-S(\rho(t)||\tau_\beta)}{\delta} \leq 0$$

which shows $\sigma(t) = -\frac{d}{dt}S(\rho(t)||\tau_\beta) \geq 0$, under thermal operations. □

When CP-divisibility breaks down, EPR can be negative and consequently the free energy of the system increases. Evidently, non-Markovianity acts as a resource and provides free energy to the system. The free energy is a monotone under thermal operations, which means that the system monotonically goes towards the thermal state. We see that non-Markovian backflow essentially drives the system away from equilibrium. Therefore, the corollary is also true for thermal operation.

6.2 Beyond thermal operations and formulation of GEPR

The entropy production rate is however, not a positive quantity for all divisible operations which are not thermal. This follows from the fact that for operations which are not thermal, the state τ_β is not a fixed point any more, and hence, we have $S(\Lambda(\rho(t))||\Lambda(\tau_\beta)) \neq S(\Lambda(\rho(t))||\tau_\beta)$. In order to investigate such situations, let us first define the notion of athermality.

Athermality: *We define athermality between the instantaneous state $\rho(t)$ and the thermal state as $\mathcal{A}(t) = \frac{1}{2}D(\rho(t),\tau_\beta)$, where $D(A,B) = ||A-B||_1$ is the trace distance between two states A and B.*

Since $\mathcal{A}(t)$ is a monotone under divisible thermal operation, it is also a witness for non-Markovian backflow. In the following Theorem, we establish a complementary relation between non-Markovianity and free energy loss.

Theorem 14: Loss of free energy and athermality obeys the following complementary relation:

$$\varDelta\mathcal{F}^{\gamma}(t) + 2\gamma\mathcal{A}^{2}(t) \leq S^{\gamma}(\rho(0)||\tau_{\beta}), \quad \forall\gamma \in (0,1] \tag{6.7}$$

Proof. : Loss of generalized free energy can be defined as

$$\varDelta\mathcal{F}^{\gamma}(t) = \left[S^{\gamma}(\rho(0)||\tau_{\beta}) - S^{\gamma}(\rho(t)||\tau_{\beta})\right].$$

From the Pinsker inequality [182] for generalised Reyni divergence [183] :

$$S^{\gamma}(A||B) \geq 2\gamma D_{T}(A,B)^{2} \quad \forall\gamma \in (0,1],$$

we get the relation (6.7). The relation naturally also holds for von-Neumann relative entropy ($\gamma \to 1$). □

Therefore it is evident that the loss of free energy $\varDelta\mathcal{F}^{\gamma}(t)$ can only decrease when the athermality of the system increases, which can only happen under non-Markovian backflow of information. This shows that the complementary relation (6.7) further bolsters the importance of non-Markovianity as a resource in various quantum thermodynamic protocols.

We now consider operations $\varLambda_{G}$, which are beyond thermal operation and generally do not possess any definite long time limit. For such operations the thermal state τ_{β} is not a fixed point any more, since the backaction of bath can produce a time-dependent shift in the Hamiltonian, or external driving Hamiltonians may also be present. Consider a general time-dependent shift in the Hamiltonian, $H \to \tilde{H}(t)$ under the evolution $\varLambda_{G}$. Consequently, the thermal state is modified to a time-dependent thermal state $\tau_{\beta}(t) = \frac{e^{-\beta\tilde{H}(t)}}{\mathcal{Z}(t)}$. We then have $\varLambda_{G}(t+\delta,t)(\tau_{\beta}(t)) = \tau_{\beta}(t+\delta)$. We define a generalized entropy production rate

(GEPR) for such evolutions as

$$\tilde{\sigma}(t) = -\frac{d}{dt}S(\rho(t)||\tau_\beta(t)). \tag{6.8}$$

Thermodynamics of open quantum systems having no definite long time limit, has been considered in several earlier works [184–187]. The generalization of entropy production rate that we provide here in this work is explicitly constructed to deal with such non-equilibrium situations. This GEPR is related to EPR by

$$\tilde{\sigma}(t) = \sigma(t) - \beta(\langle W\rangle - \langle W\rangle_{th}), \tag{6.9}$$

where $\langle W\rangle = \mathrm{Tr}[\dot{\tilde{H}}(t)\rho(t)]$ and $\langle W\rangle_{th} = \mathrm{Tr}[\dot{\tilde{H}}(t)\tau_\beta(t)]$ are the workdone by $\rho(t)$ and $\tau_\beta(t)$ respectively. The proof of (6.9) is as follows. We have $\tilde{\sigma}(t) = -\frac{d}{dt}\left[\rho(t)\ln\rho(t) - \rho(t)\ln\tau_\beta(t)\right] = \frac{d}{dt}S(t) - \beta\mathcal{J}(t) - \beta Tr[\rho(t)\dot{\tilde{H}}(t)] - \frac{d}{dt}\ln\mathcal{Z}(t) = \sigma(t) - \beta\left(\langle W\rangle - \langle W\rangle_{th}\right).$

Further, we derive a similar expression for GEPR, given by

$$\tilde{\sigma}(t) = \frac{d}{dt}\left[(S(t) - S_{th}(t)) - \beta\mathcal{W}(t)\right], \tag{6.10}$$

where $S_{th}(t)$ is the von-Neumann entropy of $\tau_\beta(t)$. The proof of (6.10) is as follows.

$$\begin{aligned} S(\rho(t)||\tau_\beta(t)) &= \mathrm{Tr}[\rho(t)\ln\rho(t)] - \mathrm{Tr}[\rho(t)\ln\tau_\beta(t)], \\ &= -(S(t) - S_{th}(t)) + \beta\mathrm{Tr}[\tilde{H}(t)(\rho(t) - \tau_\beta(t))]. \end{aligned}$$

Therefore, differentiating the above equation with respect to time, we get $\tilde{\sigma}(t) = \frac{d}{dt}\left[(S(t) - S_{th}(t)) - \beta\mathcal{W}(t)\right]$. Based on these findings, we now prove the following corollary for GEPR.

Corollary: GEPR is negatively proportional to the time rate of change of the difference between the free energies of the state $\rho(t)$ and $\tau_\beta(t)$:

$$\beta\tfrac{d}{dt}\left(\mathcal{F}(t) - \mathcal{F}_{th}(t)\right) = -\tilde{\sigma}(t).$$

Proof. : Differentiating the free energy $\mathcal{F}(t)$ with respect to time, we find

$$\beta\left(\frac{d}{dt}\mathcal{F}(t)-\langle W\rangle_{th}\right)=-\tilde{\sigma}(t).$$

The free energy of the instantaneous thermal state is $\mathcal{F}_{th}(\rho_{th}(t))=-\frac{1}{\beta}\ln\mathcal{Z}(t)$. By differentiating with respect to time we get $\frac{d}{dt}\mathcal{F}(\rho_{th}(t))=\langle W\rangle_{th}$. Hence, the modified relation between the free energy rate and the GEPR is given by

$$\beta\frac{d}{dt}\left(\mathcal{F}(t)-\mathcal{F}_{th}(t)\right)=-\tilde{\sigma}(t),$$

where $\mathcal{F}_{th}(t)=-\frac{1}{\beta}\ln\mathcal{Z}(t)$ is the free energy of the instantaneous thermal state $\tau_\beta(t)$. □

It follows that the negativity of GEPR implies that the system is free energetically going away from the instantaneous thermal state. Similar to the case of thermal operation, we establish the following complementary relation.

Corollary : A complementary relation of the form: $\varDelta\bar{\mathcal{F}}+2\mathcal{A}^2(t)\leq S(\rho(0)||\tau_\beta(0))$, exists for operations $\varLambda_G$, where $\bar{\mathcal{F}}(t)=(\mathcal{F}(\rho(t))-\mathcal{F}(\tau_\beta(t)))$ is the free energy difference between the state $\rho(t)$ and $\tau_\beta(t)$, $\mathcal{A}(t)=D_T(\rho(t)||\rho_{th}(t))$ is the instantaneous athermality and $\varDelta\bar{\mathcal{F}}(t)=(\bar{\mathcal{F}}(0)-\bar{\mathcal{F}}(t))$.

The consequences of above corollaries are rather similar to that of what we found for thermal operations. They show that non-Markovian backflow is necessary to drive the system away from its instantaneous equilibrium state and hence, is indispensable for regenerating the resource, which is the free energetic difference between the state and the thermal state. In the following section, we consider a realistic example for a central spin system, to validate our theory of GEPR.

6.3 Example of a spin bath model

Here we examine the validity of our findings for the $\varLambda_G$ operation in the backdrop of a spin-bath model. The model consists of a single spin interacting with N number of mutually non-interacting spin-half particles. The collection of non-interacting spins is considered to be the bath. This type of fermionic bath model has been of significant interest for over the past decade [188, 189] and extremely relevant for quantum computing with NV centre [190] defects within a diamond lattice.

Let the Hamiltonian corresponding to the system, bath and their interaction be given by $\tilde{H}_S, \tilde{H}_B, \tilde{H}_I$ respectively. The total Hamiltonian $\tilde{H}$ is given by,

$$\tilde{H} = \tilde{H}_S + \tilde{H}_B + \tilde{H}_I, \tag{6.11}$$

where the system, environment and interaction Hamiltonians $\tilde{H}_I$ are respectively given by

$$\begin{aligned} &\tilde{H}_S = \hbar g \omega_0 \sigma_z, \\ &\tilde{H}_B = \hbar g \tfrac{\omega}{N} \textstyle\sum_{i=1}^{N} \sigma_z^i, \\ &\tilde{H}_I = \hbar g \tfrac{\alpha}{\sqrt{N}} \textstyle\sum_{i=1}^{N} \left(\sigma_x \sigma_x^i + \sigma_y \sigma_y^i + \sigma_z \sigma_z^i \right), \end{aligned} \tag{6.12}$$

where σ_k, $k = x, y, z$ are the Pauli matrices, with the superscript 'i' stands for the i-th particle of the bath. g is a constant with the dimension of frequency, ω_0 and ω are the dimensionless parameters respectively characterizing the difference of energy levels of the system and the environment. α is the system-environment coupling strength. Utilizing the total angular momentum of the bath spin particles $J_k = \sum\limits_{i=1}^{N} \sigma_k^i$, and using the Holstein-Primakoff transformation, given by

$$J_+ = \sqrt{N} b^\dagger \left(1 - \frac{b^\dagger b}{2N} \right)^{1/2} \quad , \quad J_- = \sqrt{N} \left(1 - \frac{b^\dagger b}{2N} \right)^{1/2} b,$$

the bath and the system-environment interaction Hamiltonians can be rewritten as

$$\begin{aligned} \tilde{H}_B \; &= -\hbar g \omega \left(1 - \tfrac{b^\dagger b}{N} \right), \\ \tilde{H}_I &= 2\hbar g \alpha \left[\sigma_+ \left(1 - \tfrac{b^\dagger b}{2N} \right)^{1/2} b + \sigma_- b^\dagger \left(1 - \tfrac{b^\dagger b}{2N} \right)^{1/2} \right] \\ &\quad - \hbar g \alpha \sqrt{N} \sigma_z \left(1 - \tfrac{b^\dagger b}{N} \right). \end{aligned} \tag{6.13}$$

Here b and $b^\dagger$ are bosonic annihilation and creation operators respectively. We take the initial system-bath state as $\rho_S(0) \otimes \rho_B(0)$. The initial system qubit is considered as $\rho_S(0) = \rho_{11}(0) |1\rangle \langle 1| + \rho_{22}(0) |0\rangle \langle 0| + \rho_{12}(0) |1\rangle \langle 0| + \rho_{21}(0) |0\rangle \langle 1|$, whereas the initial environment state is taken to be a thermal state $\rho_B(0) = \exp(-\tilde{H}_B/K\tilde{T})$ with an arbitrary temperature $\tilde{T}$, where K is the Boltzmann constant. The reduced dynamics of the system state can then be calculated [191] as $\rho_S(t) = \mathrm{Tr}_B\left[\exp\left(-iHt\right)\rho_S(0)\otimes\rho_B(0)\exp\left(iHt\right)\right]$. Here

$$H = \frac{\tilde{H}}{\hbar g}, \quad t = g\tilde{t}, \quad \text{and} \quad T = \frac{K\tilde{T}}{\hbar g},$$

where H, t and T are all dimensionless quantities. Solving the global Schrödinger equation corresponding to the above mentioned Hamiltonian, we get

$$\begin{aligned}\rho_{11}(t) &= \rho_{11}(0)(1-A(t)) + \rho_{22}(0)B(t),\\ \rho_{12}(t) &= \rho_{12}(0)C(t),\end{aligned} \tag{6.14}$$

with

$$A(t) = \textstyle\sum_{n=0}^{N}(n+1)\alpha^2(1-n/2N)\left(\frac{\sin(\eta t/2)}{\eta/2}\right)^2\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},$$

$$B(t) = \textstyle\sum_{n=0}^{N} n\alpha^2(1-(n-1)/2N)\left(\frac{\sin(\eta' t/2)}{\eta'/2}\right)^2\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},$$

$$\begin{aligned}C(t) = \textstyle\sum_{n=0}^{N} & e^{-i(\Lambda-\Lambda')t/2}\left(\cos(\eta t/2) - i\frac{\theta}{\eta}\sin(\eta t/2)\right)\\ &\times\left(\cos(\eta' t/2) + i\frac{\theta'}{\eta'}\sin(\eta' t/2)\right)\frac{e^{-\frac{\omega}{T}(n/N-1)}}{Z},\end{aligned}$$

$$Z = \textstyle\sum_{n=0}^{N} e^{-\frac{\omega}{T}(n/N-1)},$$

$$\eta = 2\sqrt{\left(\omega_0 - \frac{\omega}{2N} - \alpha\sqrt{N}\left(1-\frac{2n+1}{2N}\right)\right)^2 + 4\alpha^2(n+1)(1-\frac{n}{2N})},$$

$$\eta' = 2\sqrt{\left(\omega_0 - \frac{\omega}{2N} - \alpha\sqrt{N}\left(1-\frac{2n-1}{2N}\right)\right)^2 + 4\alpha^2 n(1-\frac{(n-1)}{2N})},$$

$$\theta = 2\left(\omega_0 - \omega/2N + \alpha\sqrt{N}\left(1-\frac{2n+1}{2N}\right)\right),$$

$$\theta' = -2\left(\omega_0 - \omega/2N - \alpha\sqrt{N}\left(1 - \frac{2n-1}{2N}\right)\right),$$
$$\Lambda = -2\omega\left(1 - \frac{2n+1}{2N}\right) - \frac{\alpha}{\sqrt{N}},$$
$$\Lambda' = -2\omega\left(1 - \frac{2n-1}{2N}\right) - \frac{\alpha}{\sqrt{N}}.$$

The master equation for the reduced dynamics presented above [192], is given by $\dot{\rho}(t) = \frac{i}{\hbar}U(t)[\rho(t), \sigma_z] + \Gamma_{deph}(t)[\sigma_z\rho(t)\sigma_z - \rho(t)] + \Gamma_{dis}(t)[\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}] + \Gamma_{abs}(t)[\sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}]$, where $\sigma_\pm = \frac{\sigma_x \pm i\sigma_y}{2}$, and $\Gamma_{dis}(t), \Gamma_{abs}(t), \Gamma_{deph}(t)$ are the rates of dissipation, absorption and dephasing processes respectively, and $U(t)$ corresponds to the unitary evolution.

The Lindblad coefficients: The rates of dissipation, absorption, dephasing and the unitary evolution are, respectively, given as

$$\begin{aligned}
\Gamma_{dis}(t) &= \left[\frac{d}{dt}\frac{(A(t)-B(t))}{2} - \frac{(A(t)-B(t)+1)}{2}\frac{d}{dt}\ln(1 - A(t) - B(t))\right], \\
\Gamma_{abs}(t) &= -\left[\frac{d}{dt}\frac{(A(t)-B(t))}{2} - \frac{(A(t)-B(t)-1)}{2}\frac{d}{dt}\ln(1 - \alpha(t) - \beta(t))\right], \\
\Gamma_{deph}(t) &= \frac{1}{4}\frac{d}{dt}\left[\ln\left(\frac{1-A(t)-B(t)}{|C(t)|^2}\right)\right], \\
U(t) &= -\frac{1}{2}\frac{d}{dt}\left[\ln\left(1 + \left(\frac{C_R(t)}{C_I(t)}\right)^2\right)\right].
\end{aligned} \tag{6.15}$$

With this we have come to the end of this chapter. To summarize, we have discussed resource theory of non-Markovianity in terms of backflow of information from the backdrop of the resource theory of purity and the resource theory of thermodynamics. We have analysed the whole thing in terms of entropy production rate of the system. We have been able to characterize fully the Lindblad operators of an unital dynamics. We have also characterized the Lindblad operators corresponding to a class of thermal operation. We have formulated generalised entropy production rate and discussed the connection with non-Markovian backflow of information.

Chapter 7

Summary and conclusions

Mathematics is beautiful due to its own abstract beauty but what makes mathematics more beautiful is its enormous applications in various problems in physics. Thus mathematics and physics flourish together hand in hand. Positive linear maps on algebra of operators is such a domain of study. This study is not only beautiful from mathematical perspective, its beauty lies in the applications in several physical problems. Historically Positive linear maps acting on C^*-algebras originated by means of abstract study of Hilbert spaces. Once the idea of positive maps was used to formulate the quantum stochastic process or later to the theory of entanglement, it got flourished tremendously. This study itself has a huge challenge as the structures of positive maps are still uncharacterised even for low dimensional operator algebras. Quantum information theory provides a nice platform to explore the the structures of positive linear maps on algebra of matrices. The theory of positive linear maps is basically one of the pillars to study bipartite entanglement theory as well as the dynamics of open quantum system. In the present thesis we have studied positive linear maps from both the aspects.

We have studied bipartite entanglement theory and its connection with the theory of positive linear maps. We have introduced a novel one parameter family of positive map acting on the operators on three dimensional complex Hilbert space. We have analysed the structure of the new positive maps and discussed the indecomposability of the family of positive maps. Though positive maps provide the theoretical tool to detect bipartite entanglement, the concept of entanglement witness is handy from the experimental perspective. We have come up with a family of indecomposable entanglement witness which can detect a class of bound entangled *i.e.* PPT entangled state. Interestingly this entanglement witness is weakly optimal. We have also find the decomposition of the entanglement witness in terms of local observables. By exploiting the idea of structural physical approximation we have constructed a new bound entangled state on two qutrit system. This new bound entangled state is undetectable by several positive maps. This work leaves some open questions. We have studied the new map from algebraic perspective. The geometric characterization of this new family of maps is still open. We do not know whether this map is an extreme point in the cone of positive maps on $\mathcal{B}(\mathbb{C}^3)$. It is also not known whether this map is exposed on $\mathcal{B}(\mathbb{C}^3)$. Identification of the stable subspace [59] of the proposed family of positive map is also an open question in this direction. Moreover it will be interesting to find whether this new bound entangled state can be used some information theoretic task.

In the later part of the thesis we have concentrated on the dynamics of open quantum system. During the time evolution of a quantum system, non-Markovian information backflow from the environment to the system can be harnessed to perform several information theoretic tasks. It is known that breaking of complete positive divisibility is necessary to have the backflow of information. We are interested in constructing a formal resource theory of non-Markovianity in terms of this backflow of information. As non-Markovianity is a dynamics resource, it is quite different from the resource theories like resource theory of entanglement or coherence. Here we identify the resourceless or free operations. We have formulated the structure of the corresponding resource theory and under the assumption of small time interval approximation we have been able to show that the resource theory is convex. We have devised a measure of non-Markovianity and showed that it is lower bounded by the famous RHP measure of non-Markovianity. Moreover we have defined the idea of robustness of non-Markovianity. This measure has an operational significance. We have been able to show the connection between robustness of non-Markovianity and RHP measure.

We have exploited the convex structure of Markovian operations under small time interval approximation to construct linear witness for non-Markovianity. This question is important from the perspective of resource theory. In resource theory of entanglement geometric structure of separable states provides the pathway to construct entanglement witness. Once we have the convex and compact structure of the Markovian operations under small time interval approximation, we have shown the existence of separating hyperplane. We have constructed linear non-Markovian witness in various ways. We have used spectral decomposition of the Choi operator in one of the techniques. Interestingly we have shown that two qubit maximally entangled states, also known as, Bell states can be used to detect non-Markovianity of qubit dynamics. We have discussed the detection of non-Markovianity which can not be detected by various measures. Given a non-Markovian operation we have discussed about the existence of nearest Markovian operation. We have also found a condition for which this nearest Markovian operation is unique. This unique nearest Markovian operation can also be used to construct a linear non-Markovianity witness. Further we shed light on the geometry of Markovian operations. We have shown that the set of Markovian operations does not form a polytope. This readily gives the idea of the existence of non-linear non-Markovianty witness.

Finally we have analysed the resource theory of non-Markovianity from the perspective of resource interconversion. We have studied non-Markovianity in the backdrop of resource theory of purity and resource theory of thermodynamics. We have been able to characterize fully the Lindblad operators corresponding to unital operations which are the free operations in the resource theory of purity. We have also been able to characterize the Lindblad operators corresponding to a class of thermal operations. Exploiting the concept of entropy production rate we have been able to shown non-Markovian backflow of information is necessary to drive the system away from the equilibrium. This fact is true for both unital and thermal operations. We have formulated generalised entropy production rate and discuss its connection with non-Markovianity. We validate the results in spin-bath model.

It is important to mention here that the convex resource theory of non-Markovianity gives rise to several questions. As the existence of the nearest Markovian operation(s) corresponding to a given non-Markovian operation has been established, one can think of best Markovian approximation of a non-Markovian operation. Finding best Markovian approximation of a non-Markovian operation will be interesting. Further the existence of catalysis in some of the resource theories is a remarkable phenomenon. In the context of resource theory of non-Markovianity the question of existence of catalysis can be addressed in future.

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